Stochastic Partial Differential Equations in Fluid Mechanics
Lecture 5: Transport noise (continuation)

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Summary (and outline)

- In the previous lecture we have introduced *transport noise*
- It is motivated in various ways:
  - random perturbation of the Lagrangian motion
  - variational principles and geometric mechanics
  - small-scale action on large scale dynamics.
- We have seen that Stratonovich multiplication (namely Itô plus a corrector) is the natural choice coming from smooth approximations of the noise
- and we have found the form of the corrector, a second order elliptic operator.

(Summary and) outline

- Today we see a few elements of rigorous theory of existence and uniqueness for equations with transport noise
- Then we investigate the *eddy dissipation* scaling limit for the heat equation
- The analogous *eddy viscosity* scaling limit, for the 2D Navier-Stokes equations, has been developed recently but we onlt address the literature, in the notes.
- And finally we discuss a few elements of the 3D theory, mostly open.

Existence and uniqueness for the heat equation with transport noise

Recall $\theta(t,x)$ = temperature, $\kappa > 0$ heat diffusion constant

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

$u \cdot \nabla \theta$ = transport due to the fluid motion. When

$$u(t,x) = \sum_{k \in K} \sigma_k(x) \partial_t W_t^k$$

the correct interpretation is the Stratonovich form

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta$$

which means Itô+correction:

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta$$

Existence and uniqueness for the heat equation with transport noise

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta$$

Recall $\mathcal{L}$ is the elliptic differential operator

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sigma_k(x) \cdot \nabla (\sigma_k(x) \cdot \nabla \theta(x))$$

which can be rewritten in the form

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i (Q_{ij}(x,x) \partial_j \theta(x))$$

where

$$Q(x,y) = \mathbb{E}[W(t,x) \otimes W(t,y)] = \sum_{k \in K} \sigma_k(x) \otimes \sigma_k(y) \qquad x, y \in D.$$

Existence and uniqueness for the heat equation with transport noise

We know two very efficient methods:

1. variational,
2. semigroups.

Variational method

We limit ourselves to the ideas.

- One has to introduce a sequence of well posed approximating problems. We skip this step.
- On these approximations, one has to prove estimates independent of the approximating parameter.
- *We perform such step on the true equation, in the style of a priori estimates: we assume to have a smooth solution and see which estimates hold.*
- Such estimates provide the basis of application of the compactness method. We skip the details of this step.

Variational method, a priori estimates

If we use Stratonovich formulation (*with heat source q in the notes*)

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta$$

and we accept that the rules of calculus (being the limit of smooth noise) are the classical ones, we get

$$\begin{aligned} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 &= -2 \left\langle \theta, \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k \right\rangle + 2 \langle \theta, \kappa \Delta \theta \rangle \\ &= -2\kappa \|\nabla \theta(t)\|_{L^2}^2 \end{aligned}$$

because (recall $\operatorname{div} \sigma_k = 0$)

$$\begin{aligned} 2 \int_D \langle \theta, \sigma_k \cdot \nabla \theta \rangle &= \int_D \sigma_k(x) \cdot \nabla \theta^2(x)\, dx \\ &= -\int_D \operatorname{div} \sigma_k(x)\, \theta^2(x)\, dx = 0. \end{aligned}$$

Variational method, a priori estimates

Therefore

$$\frac{d}{dt}\left\|\theta\left(t\right)\right\|_{L^2}^2+2\kappa\left\|\nabla\theta\left(t\right)\right\|_{L^2}^2=0$$

leading to the a.s. (deterministic!) estimate

$$\left\|\theta\left(t\right)\right\|_{L^2}^2+2\kappa\int_0^t\left\|\nabla\theta\left(s\right)\right\|_{L^2}^2 ds=\left\|\theta_0\right\|_{L^2}^2 .$$

This gives us the a priori estimates

$$\begin{aligned}\sup_{t\in[0,T]}\left\|\theta\left(t\right)\right\|_{L^2}^2 &\leq C\\ \int_0^T\left\|\nabla\theta\left(s\right)\right\|_{L^2}^2 ds &\leq C.\end{aligned}$$

Variational method, a priori estimates

If we use Itô formulation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta$$

and we apply Itô formula, we get

$$\begin{aligned} d \|\theta(t)\|_{L^2}^2 &= -2 \sum_{k \in K} \langle \theta, (\sigma_k \cdot \nabla \theta) \rangle dW^k + 2 \langle \theta, (\kappa \Delta + \mathcal{L}) \theta \rangle dt \\ &\quad + \sum_{k \in K} \|\sigma_k \cdot \nabla \theta\|_{L^2}^2 dt \\ &= -2\kappa \|\nabla \theta(t)\|_{L^2}^2 - 2 \frac{1}{2} \int_D \sum_{ij} Q(x,x) \partial_i \theta \partial_j \theta dx dt \\ &\quad + \sum_{k \in K} \int_D \sum_{ij} \sigma_k^i(x) \partial_i \theta \sigma_k^j(x) \partial_j \theta dx dt \end{aligned}$$

and get the same as above. *At the level of energy estimates, the Itô term and the corrector completely balance each other.*

Semigroup method

Consider the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \, \partial_t W^k = (\kappa \Delta + \mathcal{L}) \, \theta.$$

Call: $H = L^2(D)$, $V = W_0^{1,2}(D)$, $D(A) = W^{2,2}(D) \cap V$,
$A : D(A) \subset H \to H$

$$A\theta = (\kappa \Delta + \mathcal{L}) \, \theta$$

e^{tA}, $t \geq 0$, the analytic semigroup generated by A. Then

$$\theta(t) = e^{tA}\theta_0 + \sum_{k \in K} \int_0^t e^{(t-s)A} (\sigma_k \cdot \nabla \theta(s)) \, dW_s^k.$$

These equations are not trivial because there is $\nabla\theta$ on the right-hand-side and thus iteration (for a fixed point theorem) requires that also the left-hand-side accepts a gradient.

Semigroup method. Notion of solution

Definition

A stochastic process

$$\theta \in C_{\mathcal{F}}\left([0,T];H\right) \cap L^2_{\mathcal{F}}\left(0,T;V\right)$$

is a mild solution if the following identity holds

$$\theta(t) = e^{tA}\theta_0 - \sum_{k \in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla \theta(s)\, dW_s^k$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

One can give a definition of weak solution and prove equivalence.

Semigroup method. Main result

Consider the equation (here let us add the source q)

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta + q$$

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k \in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla \theta(s)\, dW_s^k$$

Theorem

For every $\theta_0 \in H$ and $q \in L^2(0, T; H)$, there exists one and only one (weak or mild) solution.

Semigroup method. General equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = \sum_{i,j=1}^{d} \partial_j (a_{ij}(x) \partial_i \theta) + q$$

where $a_{i,j}$ is strongly elliptic and sufficiently regular so that the operator $A\theta = \sum_{i,j=1}^{d} \partial_j (a_{i,j}(x) \partial_i \theta)$ generates an analytic semigroup. The notions of solutions are the same.

Theorem

Assume the exists $\eta < 1$ such that

$$\frac{1}{2} \sum_{k \in K} (\sigma_k(x) \cdot \xi)^2 \leq \eta \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j$$

for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. Then, for every $\theta_0 \in H$ and $q \in L^2(0, T; H)$, there exists one and only one (weak or mild) solution.

Super-parabolicity and Stratonovich

The super-parabolicity condition

$$\frac{1}{2}\sum_{k\in K}\left(\sigma_k\left(x\right)\cdot\xi\right)^2\leq\eta\sum_{i,j=1}^{d}a_{ij}\left(x\right)\xi_i\xi_j\qquad\eta<1$$

is always true when

$$a_{ij}\left(x\right)=\kappa\delta_{ij}+\frac{1}{2}Q_{ij}\left(x,x\right)$$

$$Q_{ij}\left(x,x\right)=\sum_{k\in K}\sigma_k^i\left(x\right)\sigma_k^j\left(x\right).$$

- Zakai equation of filtering requires super-parabolicity.
- Stratonovich is always well posed.

Proof of well posedness: auxiliary variables

$$\theta(t) = e^{tA}\theta_0 - \sum_{k\in K}\int_0^t e^{(t-s)A}\sigma_k \cdot \nabla\theta(s)\, dW_s^k$$

$$v_h(t) = \sigma_h \cdot \nabla e^{tA}\theta_0 - \sum_{k\in K}\int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k$$

for $h \in K$. Equivalence by:

$$\begin{aligned} v_k(t) &:= \sigma_k \cdot \nabla\theta(t) \\ v(t) &:= (v_k(t))_{k\in K} \end{aligned}$$

$$\theta(t) = e^{tA}\theta_0 + \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} q(s)\, ds - \sum_{k\in K}\int_0^t e^{(t-s)A} v_k(s)\, dW_s^k$$

Proof of well posedness: auxiliary variables

Consider the space X_T of vectors $(v_k(\cdot))_{k\in K}$ such that $v_k \in L^2_{\mathcal{F}}(0,T;H)$

$$\|v\|_T^2 := \sum_{h\in K} \mathbb{E}\int_0^T \|v_h(t)\|_H^2\, dt.$$

It is a Hilbert space and $\|v\|_T$ is the induced norm. Consider

$$v_h(t) = \sigma_h \cdot \nabla e^{tA}\theta_0 - \sum_{k\in K}\int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k$$

for $h \in K$.

Theorem

There exists a unique solution $(v_k(\cdot))_{k\in K} \in X_T$.

Proof of well posedness: auxiliary variables

Choose a number $\epsilon > 0$ so small that $\eta(1+\epsilon) < 1$. Consider the map Γ defined on X_T as

$$(\Gamma v)_h(t) := w_h(t) + \sum_{k\in K} \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k$$

$h \in K$, where $w_h(t) := \sigma_h \cdot \nabla e^{tA}\theta_0$. We have

$$\begin{aligned}\|\Gamma v\|_T^2 \leq& \left(1+\frac{4}{\epsilon}\right) \sum_{h\in K} \int_0^T \mathbb{E}\left[\|w_h(t)\|_{L^2}^2\right] dt \\ &+ (1+\epsilon) \sum_{h\in K} \int_0^T \mathbb{E}\left[\left\|\sum_{k\in K} \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k\right\|_{L^2}^2\right] dt\end{aligned}$$

$$
(1+\epsilon)\sum_{h\in K}\int_0^T \mathbb{E}\left[\left\|\sum_{k\in K}\int_0^t \sigma_h\cdot\nabla e^{(t-s)A} v_k(s)\, dW_s^k\right\|_{L^2}^2\right] dt
$$

$$
= (1+\epsilon)\sum_{h\in K}\int_0^T\int_s^T \mathbb{E}\left[\sum_{h\in K}\left\|\sigma_h\cdot\nabla e^{(t-s)A} v_k(s)\right\|_{L^2}^2\right] dtds
$$

$$
\leq -2\eta(1+\epsilon)\sum_{k\in K}\int_0^T\int_s^T\left\langle A e^{(t-s)A} v_k(s), e^{(t-s)A} v_k(s)\right\rangle dtds
$$

$$
\leq \eta(1+\epsilon)\|v\|_T^2 \qquad \left(-2\int_s^T\left\langle A e^{(t-s)A} h, e^{(t-s)A} h\right\rangle dt \leq \|h\|_H^2\right)
$$

$$
\left\|\Gamma v' - \Gamma v''\right\|_T^2 \leq \eta(1+\epsilon)\left\|v' - v''\right\|_T^2 .
$$

Since $\eta(1+\epsilon) < 1$, Γ is a contraction (independently of T).

Equation for the average

Defined

$$\Theta(t,x) := \mathbb{E}[\theta(t,x)].$$

and assumed θ_0, q deterministic,

Theorem

$\Theta(t,x)$ is a (weak or mild) solution of the deterministic equation

$$\begin{aligned} \partial_t \Theta &= (\kappa\Delta + \mathcal{L})\Theta + q \\ \Theta|_{t=0} &= \theta_0. \end{aligned}$$

When the random temperature is close to its mean

We ask here: when θ is close to Θ? *Main assumption*: define $\varepsilon_{Q,\kappa} \geq 0$ as the smallest number such that

$$\begin{aligned} & \int \int v(x)^T Q(x,y) v(y)\, dxdy \\ \leq \quad & \varepsilon_{Q,\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x) v(x) \right) dx \end{aligned}$$

for all $v \in L^2(D, \mathbb{R}^d)$.
We shall need

$$\varepsilon_{Q,\kappa} \text{ small.}$$

Below we shall interpret this assumption. Notice it is given only in terms of Q and κ.

When the random temperature is close to its mean

$$
\begin{aligned}
\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k &= (\kappa \Delta + \mathcal{L}) \theta + q \\
\partial_t \Theta &= (\kappa \Delta + \mathcal{L}) \Theta + q
\end{aligned}
$$

with the same θ_0. Call $C_\infty (T, \theta_0, q) > 0$ a constant such that

$$
\sup_{s \in [0,T]} \mathbb{E} \|\theta(s)\|_\infty^2 \leq C_\infty (T, \theta_0, q).
$$

Theorem

For every $\phi \in L^2(D)$,

$$
\mathbb{E} \left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \leq \varepsilon_{Q,\kappa} \|\phi\|_{L^2}^2 C_\infty (T, \theta_0, q).
$$

Proof

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k\in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k.$$

Here $e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds$ is precisely $\Theta(t)$, hence

$$\theta(t) - \Theta(t) = -\sum_{k\in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k.$$

$$\langle \theta(t) - \Theta(t), \phi \rangle = \sum_{k\in K} \int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla\theta e^{(t-s)A}\phi \right\rangle dW_s^k.$$

Then (here we take advantage of the cancellations of Itô integrals)

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] = \sum_{k\in K} \mathbb{E} \int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla e^{(t-s)A}\phi \right\rangle^2 ds.$$

Proof

Write $\phi_{t,s} := e^{(t-s)A}\phi$. Then

$$\begin{aligned}
&\sum_{k\in K}\left\langle \theta\left(s\right),\sigma_{k}\cdot\nabla\phi_{t,s}\right\rangle^{2} \\
= &\sum_{k\in K}\int\int\theta\left(s,x\right)\theta\left(s,y\right)\sigma_{k}\left(x\right)\cdot\nabla\phi_{t,s}\left(x\right)\sigma_{k}\left(y\right)\cdot\nabla\phi_{t,s}\left(y\right)dxdy \\
= &\int\int\theta\left(s,y\right)\nabla\phi_{t,s}\left(y\right)^{T}Q\left(x,y\right)\nabla\phi_{t,s}\left(x\right)\theta\left(s,x\right)dxdy \\
\leq &-\varepsilon_{Q,\kappa}\left\|\theta\left(s\right)\right\|_{\infty}^{2}\left\langle Ae^{(t-s)A}\phi,e^{(t-s)A}\phi\right\rangle .
\end{aligned}$$

Proof

Therefore

$$\begin{aligned}
& \mathbb{E}\left[\langle\theta(t)-\Theta(t),\phi\rangle^{2}\right] \\
\leq\ & \varepsilon_{Q,\kappa}C_{\infty}(T,\theta_{0},q)\int_{0}^{t}\left\langle(-A)e^{(t-s)A}\phi,e^{(t-s)A}\phi\right\rangle ds \\
=\ & \varepsilon_{Q,\kappa}C_{\infty}(T,\theta_{0},q)\int_{0}^{t}\frac{d}{ds}\left\|e^{(t-s)A}\phi\right\|_{L^{2}}^{2}ds \\
\leq\ & \varepsilon_{Q,\kappa}C_{\infty}(T,\theta_{0},q)\left\|\phi\right\|_{L^{2}}^{2}.
\end{aligned}$$

Relevance of the result. An example

Infinite channel

$$D = \mathbb{R} \times [-1, 1]$$

$$\theta(x_1, \pm 1) = \sigma_k(x_1, \pm 1) = 0 \qquad \text{for every } x_1 \in \mathbb{R},\ k \in K.$$

The theoretical results are similar to those above. In addition, let us consider the *stationary deterministic profile* for a given $q = q(x)$, element of H: we have to solve

$$A\Theta_{st} + q = 0$$

$$\Theta_{st} = -A^{-1}q.$$

Relevance of the result. An example

In practice, assume that in a region $x \in [-L, L] \times [-1, 1]$ the function $q(x)$ is equal to a constant q, and both the stationary solution $\Theta_{st}(x)$ and $Q(x, x)$ depend only on the vertical direction $y \in [-1, 1]$ and they are symmetric with respect to $y = 0$. The equation

$$\mathrm{div}\left(\left(\kappa I + \frac{1}{2} Q(x, x)\right) \nabla \Theta_{st}(x)\right) = -q(x)$$

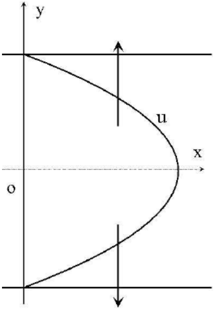
becomes

$$\partial_y \left(\left(\kappa + Q_{22}(y)\right) \partial_y \Theta_{st}(y)\right) = -q.$$

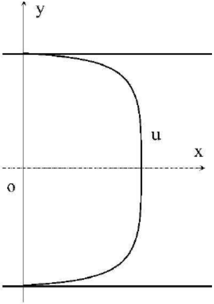
Relevance of the result. An example

The solution of the previous equation is

$$\Theta_{st}(y) = -\int_{-1}^{y} \frac{qs}{\kappa + Q_{22}(s)} ds.$$



(a) Laminar flow



(b) Turbulent flow

Concerning the assumption

Recall

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] \leq \varepsilon_{Q,\kappa} \|\phi\|_{L^2}^2 C_\infty(T, \theta_0, q)$$

where $\varepsilon_{Q,\kappa}$ is given by

$$\begin{aligned} & \int\int v(x)^T Q(x,y) v(y)\, dxdy \\ \leq \quad & \varepsilon_{Q,\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x) v(x)\right) dx. \end{aligned}$$

The question is:

When is $\varepsilon_{Q,\kappa}$ very small?

The assumption for domains without boundary

- When D "has no boundary" (torus or full space), we may take $Q(x,y)$ of very special form (e.g. Kraichnan noise, including Kolmogorov 41).
- In this case it is easy to make examples where

$$\int\int v(x)^T Q(x,y) v(y)\, dxdy \text{ is very small} \qquad (\sim \text{operator norm})$$

$$\int \frac{1}{2} v(x)^T Q(x,x) v(x)\, dx \text{ is very large} \qquad (\sim \text{operator trace}).$$

- Hence the following holds with small $\varepsilon_{Q,\kappa}$ (we do not need the term $\int \kappa |v(x)|^2 dx$)

$$\int\int v(x)^T Q(x,y) v(y)\, dxdy \leq \varepsilon_{Q,\kappa} \int \frac{1}{2} v(x)^T Q(x,x) v(x)\, dx.$$

The assumption for domains with boundary

- When D has a boundary (our case), Q degenerates at the boudary $(\sigma_k|_{\partial D} = 0)$.
- Then the term $\int \frac{1}{2} v(x)^T Q(x,x) v(x) \, dx$ does not help so much.
- We have examples which satisfy

$$\int \int v(x)^T Q(x,y) v(y) \, dxdy \leq \varepsilon_{Q,\kappa} \int \kappa |v(x)|^2 \, dx$$

with very small $\varepsilon_{Q,\kappa}$.

The 3D case. Passive magnetic field

The equations for a magnetic field M in a fluid u are

$$\partial_t M + u \cdot \nabla M = \eta \Delta M + M \cdot \nabla u.$$

Similarly to the scalar case, we model u by a white noise, with the Stratonovich interpretation:

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt + \sum_{k \in K} M \cdot \nabla \sigma_k \circ dW_t^k.$$

The equation can be written as

$$dM = (\eta \Delta + \mathcal{L}) M dt + \text{Itô terms}$$

for a suitable second order differential operator $\mathcal{L}$. And $\overline{M} := \mathbb{E}[M]$ satisfies

$$\partial_t \overline{M} = (\eta \Delta + \mathcal{L}) \overline{M}.$$

The 3D case. Passive magnetic field

- Thus, as above, the question arises whether $\mathbb{E}\left[\left\langle M\left(t\right)-\overline{M}\left(t\right),\phi\right\rangle^{2}\right]$ is small.
- There exists the following conjecture:
  F. Krause, K.-H. Rädler, Mean Field Magnetohydrodynamics, 1980, page 12: "homogeneous isotropic mirror symmetric turbulence only influences the decay rate of the mean magnetic fields, which is enhanced in almost all cases of physical interest."
- Unfortunately, this problem remains open. Let us explain why.

Define

$$B_k M = M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M$$

Then the corrector is

$$\frac{1}{2} \sum_{k \in K} B_k B_k M.$$

We have

$$\begin{aligned} B_k B_k M &= (B_k M) \cdot \nabla \sigma_k - \sigma_k \cdot \nabla (B_k M) \\ &= (M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M) \cdot \nabla \sigma_k - \sigma_k \cdot \nabla (M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M) \end{aligned}$$

$$\begin{aligned} &= (M \cdot \nabla \sigma_k) \cdot \nabla \sigma_k - (\sigma_k \cdot \nabla M) \cdot \nabla \sigma_k \\ &\quad - \sigma_k \cdot \nabla (M \cdot \nabla \sigma_k) + \sigma_k \cdot \nabla (\sigma_k \cdot \nabla M). \end{aligned}$$

The 3D case. Passive magnetic field. The corrector

Lemma

$$\frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}_{scalar} M - \sum_{i,j}\left(\sum_{k\in K}\sigma_k^i \partial_j \sigma_k\right)\partial_i M_j$$

$$+\frac{1}{2}\sum_j\left(\sum_i\sum_{k\in K}\left(\partial_j\sigma_k^i\partial_i\sigma_k - \sigma_k^i\partial_i\partial_j\sigma_k\right)\right)M_j.$$

Lemma

Assume the noise is space-homogeneous, $Q(x,y) = Q(x-y)$. *Then*

$$\frac{1}{2}\sum_j\left(\sum_i\sum_{k\in K}\left(\partial_j\sigma_k^i\partial_i\sigma_k - \sigma_k^i\partial_i\partial_j\sigma_k\right)\right)M_j = 0.$$

The 3D case. Passive magnetic field. The corrector

Lemma

If the noise is space-homogeneous, then

$$\frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}_{scalar} M - \sum_j \partial_j Q(0) \cdot \nabla M_j$$

where $\partial_j Q(0)$ *is the matrix with entries* $(\partial_j Q_{\alpha,i})(0)$*. In the particular case when*

$$Q(-x) = Q(x)$$

(mirror symmetry) then $\partial_j Q(0) = 0$ *and thus*

$$\frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}_{scalar} M.$$

- Thus we see that the Itô-Stratonovich corrector is similar to the scalar case, at least under suitable assumptions.
- *The problem is that we need estimates on* M, in order to prove that $\langle M(t), \phi \rangle - \langle \overline{M}(t), \phi \rangle$ is small.
- These estimates, at present, are not available. The difficulty is due to the term

$$M \cdot \nabla \sigma_k.$$

- Let us see for instance what happens to energy estimates.

The 3D case. Passive magnetic field

$$dM + \sum_{k\in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt + \sum_{k\in K} M \cdot \nabla \sigma_k \circ dW_t^k$$

$$\begin{aligned} & d \left\| M(t) \right\|_{L^2}^2 + 2 \sum_{k\in K} \langle \sigma_k \cdot \nabla M, M \rangle \circ dW_t^k \\ = & -2\eta \left\| \nabla M(t) \right\|_{L^2}^2 dt + 2 \sum_{k\in K} \langle M \cdot \nabla \sigma_k, M \rangle \circ dW_t^k \end{aligned}$$

$$\langle \sigma_k \cdot \nabla M, M \rangle = 0$$

but

$$\langle M \cdot \nabla \sigma_k, M \rangle \neq 0.$$

If we consider the reduced model

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt$$

we can prove bounds on M and deduce that

$$\langle M(t), \phi \rangle - \langle \overline{M}(t), \phi \rangle$$

is small in mean square.
The physical meaning of this assumption, or some extensions, are under investigation.

The 3D case. Navier-Stokes equations. Only transport noise

Consider, on the 3D torus, the vorticity equation with noise only in the transport component:

$$\partial_t \omega + u \cdot \nabla \omega + P\left(u' \circ \nabla \omega\right) = \Delta \omega + \omega \cdot \nabla u.$$

with noise u' of the form

$$u'(t,x) = \sum_k \sigma_k(x)\, \partial_t W_t^k$$

- Notice the projection in $P\left(u' \circ \nabla \omega\right)$, necessary for compatibility, but source of great technical difficulties (the Itô-Stratonovich corrector is a nonlocal differential operator).
- Call ω the unique local solution, for $\omega_0 \in H$ (the space L^2 with usual conditions).

The 3D case. Navier-Stokes equations. Only transport noise

Theorem

Given $T, R_0, \epsilon > 0$ *there exists* $(\sigma_k)_{k \in K}$ *with the following property: for every initial condition* $\omega_0 \in H$ *with* $\|\omega_0\|_H \leq R_0$*, the 3D Navier-Stokes equations with transport noise (and viscosity* $=$ *1) has a global unique solution on* $[0, T]$*, up to probability* ϵ*.*

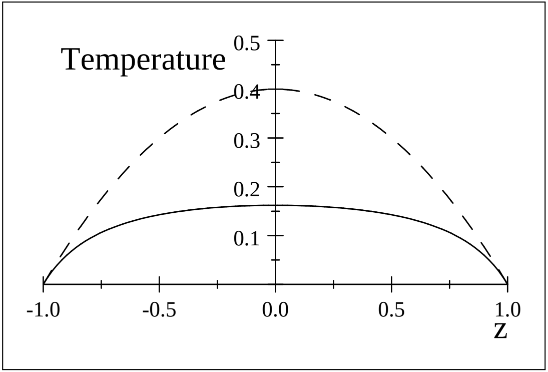
Summary

- In this chapter we discuss transport noise. Transport-stretching type in 3D is less understood.
- It introduces, by Wong-Zakai limit, an auxiliary elliptic operator.
- In the case of heat transport it proves the property of eddy dissipation.
- Similar ideas may be applied to the internal structure of the fluid, by a large/small scale analysis and stochastic modeling of small scales.
- In 2D it explains eddy viscosity: turbulence enhances the viscosity of the fluid itself

Summary

- In 3D, just transport noise (no stretching noise): it improves the theory of 3D Navier-Stokes equations, delaying the blow-up of smooth solutions.
- Deep research is needed to understand the case of transport-stretching noise.
- Heurisitc remark:
  - we started from additive perturbations motivated by the roughness of boundaries
  - additive noise in the small scales lead to multiplicative transport noise in the large scales
  - transport noise has a better regularizing power.
- At the end it seems that *it is the additive noise at small scales which regularizes*!

Thank you!



Dashed parabolic profile: $Q = 0$. Solid-line profile: large Q.