Stochastic Partial Differential Equations in Fluid Mechanics
Lecture 4: Transport noise

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Summary and outline

- In the first lecture we have discussed the origin of *noise from boundary perturbations.*
- It was an additive noise, or later on a *state-dependent noise* to account for variability of mean flow.
- Today we start investigation of *transport noise*
- discussing its physical origin from large-small scale decomposition
- and its consequences on turbulence theory.

Simplified dynamics near the boundary

Let us oversimplify the fluid dynamics near the boundary:

$$\begin{aligned}\partial_t u + \nabla p &= \nu \Delta u - \frac{1}{\epsilon} u + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k \\ \operatorname{div} u &= 0 \\ u|_{\partial D} &= 0\end{aligned}$$

This is Stokes model, strongly incorrect in itself for turbulent fluids, but complemented by the creation of eddies/vortices (the term $\frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k$) and an extra-dissipation term of friction type ($-\frac{1}{\epsilon} u$) to compensate the extra input of energy (in the average) due to the noise.

Scaling limit of the previous model

$$\partial_t u^\epsilon + \nabla p^\epsilon = \nu \Delta u^\epsilon - \frac{1}{\epsilon} u^\epsilon + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k$$

Recalling $A = \nu P \Delta$,

$$u^\epsilon (t) = e^{t\left(A - \frac{1}{\epsilon}\right)} u_0 + \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{(t-s)\left(A - \frac{1}{\epsilon}\right)} \sigma_k dW_s^k .$$

Alternatively, we may rewrite by integration by parts (Chapter 1).

Let us introduce two notations:

$$\begin{aligned} W^{\epsilon}(t,x) &= \int_0^t u^{\epsilon}(s,x)\,ds \\ W(t,x) &= \sum_{k\in K} \sigma_k(x)\, W_t^k. \end{aligned}$$

Then

$$\begin{aligned} W^{\epsilon}(t) &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t \int_0^s e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k ds \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t \int_r^t e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k ds dW_r^k \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t \left(A-\frac{1}{\epsilon}\right)^{-1}\left[e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}-1\right]\sigma_k dW_r^k \end{aligned}$$

Recall:

$$\begin{aligned} W^{\epsilon}(t,x) &= \int_0^t u^{\epsilon}(s,x)\,ds \\ W(t,x) &= \sum_{k\in K}\sigma_k(x)\,W_t^k \end{aligned}$$

$$\begin{aligned} W^{\epsilon}(t) &= \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}\sum_{k\in K}\int_0^t e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k \\ &\quad -\frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1} W(t)\,. \end{aligned}$$

Yosida approximations in semigroup theory:

$$\lim_{\lambda\to\infty}\lambda(\lambda-A)^{-1}h=h \qquad \text{for every } h\in H.$$

Summarizing: Stokes converges to white noise

$$\partial_t u^\epsilon + \nabla p^\epsilon = \nu \Delta u^\epsilon - \frac{1}{\epsilon} u^\epsilon + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k$$

$$\begin{aligned} W^\epsilon (t, x) &= \int_0^t u^\epsilon (s, x) \, ds \\ W (t, x) &= \sum_{k \in K} \sigma_k (x) W_t^k . \end{aligned}$$

Lemma

$$\lim_{\epsilon \to 0} \mathbb{E} \left[\| W^\epsilon (t) - W (t) \|_H^2 \right] = 0.$$

Temperature (as a passive scalar) in a turbulent fluid

$\theta(t,x)$ = temperature, $\kappa > 0$ heat diffusion constant

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

$u \cdot \nabla \theta$ = transport due to the fluid motion. If

$$u = u^{\epsilon} \text{ above}$$

and we take the limit $\epsilon \to 0$, and we apply the heuristics of Wong-Zakai result, we find the model

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta$$

where the symbol $\circ$ stands for the Stratonovich operation.

Eddy diffusion

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta$$

In Itô form:

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta \tag{1}$$

$\mathcal{L}$ suitable second order elliptic differential operator. Mean temperature profile:

$$\Theta (t, x) = \mathbb{E} [\theta (t, x)]$$

$$\partial_t \Theta = (\kappa \Delta + \mathcal{L}) \Theta.$$

Turbulent diffusion increases the original diffusion, the so called *eddy diffusion*.

A Wong-Zakai result

Key to the previous slide is the emergence of the additional operator $\mathcal{L}$; we feel we need to justify it, at least heuristically. Compare the equations

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \kappa \Delta \theta^\epsilon$$

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta$$

with

$$\theta^\epsilon|_{t=0} = \theta|_{t=0} = \theta_0 \in L^\infty(D)$$

$$u^\epsilon(t) = \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \sigma_k dW_s^k.$$

Both have unique weak solutions θ^ϵ and θ.

A Wong-Zakai result

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \kappa \Delta \theta^\epsilon$$

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \partial_t W^k = (\kappa \Delta + \mathcal{L}) \theta$$

Theorem

If $\sigma_k \in D(A)$, $\phi \in C^\infty(D)$, then, for every $t \geq 0$,

$$\lim_{\epsilon \to 0} \langle \theta^\epsilon(t), \phi \rangle = \langle \theta(t), \phi \rangle$$

in probability, with

$$(\mathcal{L}\theta)(x) = \sum_{k \in K} \sigma_k(x) \cdot \nabla (\sigma_k(x) \cdot \nabla \theta(x)).$$

Idea of the proof. Technical a priori estimates

Several technical estimates on θ^{ϵ} are needed. Example:

$$\left\|\theta^{\epsilon}\left(t\right)\right\|_{L^{2}}^{2}+2\kappa\int_{0}^{t}\left\|\nabla\theta^{\epsilon}\left(s\right)\right\|_{L^{2}}^{2}ds=\left\|\theta_{0}\right\|_{L^{2}}^{2}$$

$$\left\|\theta^{\epsilon}\left(t\right)\right\|_{\infty}\leq\left\|\theta_{0}\right\|_{\infty}.$$

These are independent of ϵ and even deterministic. Others are more difficult, ϵ-dependent, and omitted here.

Idea of the proof. Weak formulation and partition of the interval

Let $\pi_\epsilon = (t_i^\epsilon)$ be a partition of $[0, T]$. Apply it to the weak formulation:

$$\int_0^t \langle u^\epsilon (s) \cdot \nabla\phi, \theta^\epsilon (s) \rangle \, ds = \sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \langle u^\epsilon (s) \cdot \nabla\phi, \theta^\epsilon (s) \rangle \, ds.$$

Assume noise of dimension 1:

$$u^\epsilon (t, x) = \sigma (x) \, \xi_t^\epsilon$$

where

$$W_t^\epsilon := \int_0^t \xi^\epsilon (s) \, ds \to W_t.$$

Then

$$\int_{t_i}^{t_{i+1}} \langle u^\epsilon(s) \cdot \nabla\phi, \theta^\epsilon(s)\rangle \, ds$$
$$= \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla\phi, \theta^\epsilon(s)\rangle \, \xi_s^\epsilon ds$$
$$= \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla\phi, \theta^\epsilon(t_i)\rangle \, \xi_s^\epsilon ds + \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla\phi, (\theta^\epsilon(s) - \theta^\epsilon(t_i))\rangle \, \xi_s^\epsilon ds$$
$$= \langle \sigma \cdot \nabla\phi, \theta^\epsilon(t_i)\rangle \left(W_{t_{i+1}}^\epsilon - W_{t_i}^\epsilon\right) + \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla\phi, (\theta^\epsilon(s) - \theta^\epsilon(t_i))\rangle \, \xi_s^\epsilon ds.$$

The sum over the partition of the first term converge to the Itô integral $\int_0^t \langle \sigma \cdot \nabla\phi, \theta(s)\rangle \, dW_s$.

More difficult is to understand the limit of

$$\sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \left\langle \sigma \cdot \nabla \phi, \left(\theta^{\epsilon}(s) - \theta^{\epsilon}(t_i) \right) \right\rangle \xi_s^{\epsilon} ds.$$

Use

$$\left\langle \psi, \theta^{\epsilon}(s) - \theta^{\epsilon}(t_i) \right\rangle - \int_{t_i}^{s} \left\langle \sigma \cdot \nabla \psi, \theta^{\epsilon}(r) \right\rangle \xi_r^{\epsilon} dr = \int_{t_i}^{s} \left\langle \kappa \Delta \psi, \theta^{\epsilon}(r) \right\rangle dr$$

with

$$\psi = \sigma \cdot \nabla \phi.$$

$$\sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, (\theta^\epsilon (s) - \theta^\epsilon (t_i)) \rangle \xi_s^\epsilon ds$$

$$= \sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon (r) \rangle \xi_r^\epsilon \xi_s^\epsilon dr ds$$
$$+ \sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \langle \kappa \Delta (\sigma \cdot \nabla \phi), \theta^\epsilon (r) \rangle dr \right) \xi_s^\epsilon ds.$$

We have

$$\lim_{\epsilon \to 0} \sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \langle \kappa \Delta (\sigma \cdot \nabla \phi), \theta^\epsilon (r) \rangle dr \right) \xi_s^\epsilon ds = 0.$$

It remains:

$$\sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(r \right) \right\rangle \xi_r^{\epsilon} \xi_s^{\epsilon} drds$$

which, as above,

$$= \sum_{t_i \leq t} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(t_i \right) \right\rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{\epsilon} \xi_s^{\epsilon} drds$$
$$+ \text{ remainder (which goes to zero).}$$

Let us go back to the multidimensional noise:

$$\sum_{k\in K}\sum_{t_i\leq t}\langle\sigma_k\cdot\nabla\left(\sigma_{k'}\cdot\nabla\phi\right),\theta^{\epsilon}\left(t_i\right)\rangle\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\xi_r^{k,\epsilon}\xi_s^{k',\epsilon}drds.$$

The quadratic variation property

$$\lim_{\epsilon\to 0}\sum_{t_i\leq t}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\xi_r^{k,\epsilon}\xi_s^{k',\epsilon}drds\to\frac{1}{2}\delta_{k,k'}t$$

implies

$$\longrightarrow\frac{\delta_{k,k'}}{2}\int_0^t\langle\sigma_k\cdot\nabla\left(\sigma_{k'}\cdot\nabla\phi\right),\theta\left(s\right)\rangle ds.$$

End of the proof

Summarizing, in the weak sense,

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_0^t u^\epsilon(s) \cdot \nabla \theta^\epsilon(s)\, ds \\ = \;& \sum_{k \in K} \int_0^t \sigma_k \cdot \nabla \theta dW_s^k + \frac{1}{2} \sum_{k \in K} \int_0^t (\sigma_k \cdot \nabla \sigma_k \cdot \nabla)\, \theta(s)\, ds. \end{aligned}$$

Thus the Wong-Zakai (or Stratonovich) corrector is

$$(\mathcal{L}\theta)(x) = \sum_{k \in K} \sigma_k(x) \cdot \nabla (\sigma_k(x) \cdot \nabla \theta(x)).$$

Divergence form of the operator

$$(\mathcal{L}\theta)(x) = \sum_{k\in K} \sigma_k(x) \cdot \nabla \left(\sigma_k(x) \cdot \nabla\theta(x)\right).$$

Componentwise we can write

$$(\mathcal{L}\theta)(x) = \sum_{k\in K} \sum_{i,j=1}^{d} \sigma_k^i(x)\, \partial_i \left(\sigma_k^j(x)\, \partial_j\theta(x)\right).$$

Since $\sum_{i=1}^{d} \partial_i \sigma_k^i(x) = 0$, we deduce also

$$(\mathcal{L}\theta)(x) = \sum_{i,j=1}^{d} \partial_i \left(\left(\underbrace{\sum_{k\in K} \sigma_k^i(x)\, \sigma_k^j(x)}_{?} \right) \partial_j\theta(x) \right).$$

Covariance (matrix-) function of the noise

$$Q(x,y) = \mathbb{E}\left[W(t,x)\otimes W(t,y)\right] \qquad x,y\in D$$

$$Q(x,y) = \sum_{k\in K}\sigma_k(x)\otimes\sigma_k(y).$$

Therefore we have found

$$(\mathcal{L}\theta)(x) = \sum_{i,j=1}^{d}\partial_i\left(Q_{ij}(x,x)\,\partial_j\theta(x)\right).$$

Ellipticity:

$$\sum_{i,j=1}^{d} Q_{ij}(x,x)\,\xi_i\xi_j = \mathbb{E}\left[\left|W(t,x)\cdot\xi\right|^2\right] \geq 0$$

for all $\xi = (\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$.

Additional stochastic transport in the Navier-Stokes equations

- Stochastic transport of passive scalars (the topic described in the previous section) is well known in the literature.
- On the contrary, we now introduce an analogous idea for the *internal modeling of a fluid*, which is less common and still debated.
- In some cases however it leads to results observed in the real world, hence it deserves to be investigated.

Large and small scales

$$u(t,x) = \overline{u}(t,x) + u'(t,x)$$

- $\overline{u}(t,x)$ containing most of the large scales
- $u'(t,x)$ mostly related to the small scales.
- A precise subdivision is impossibile, due to the multiscale nature of the problem.

Large and small scales by projections

Attempt of precise subdivision by means of projections:

1. given (e_n) c.o.s. of H, associated projections π_n, define

$$\overline{u}(t) = \pi_n u(t)$$

2. given mollifiers $\theta_\epsilon(x) = \epsilon^{-d}\theta\left(\epsilon^{-1}x\right)$, define

$$\overline{u}(t) = \theta_\epsilon * u(t).$$

Drawback: difficult interlaced equations.

Large and small scales by a simplified system

Consider the Navier-Stokes type system

$$\begin{aligned} \partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\ \partial_t u' + \left(\overline{u} + u'\right) \cdot \nabla u' + \nabla p' &= \nu \Delta u' + f' \\ \operatorname{div} \overline{u} &= \operatorname{div} u' = 0, \qquad \overline{u}|_{\partial D} = u'|_{\partial D} = 0 \\ \overline{u}(0) &= \overline{u}_0, \qquad u'(0) = u'_0. \end{aligned}$$

It is equivalent, by $u = \overline{u} + u'$, $p = \overline{p} + p'$, to the original equation

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ \operatorname{div} u &= 0, \qquad u|_{\partial D} = 0, \qquad u(0) = u_0 \end{aligned}$$

when

$$\begin{aligned} f &= \overline{f} + f' \\ u_0 &= \overline{u}_0 + u'_0. \end{aligned}$$

Small scales are quite concentrated in a region near the boundary, the large scales are active everywhere.

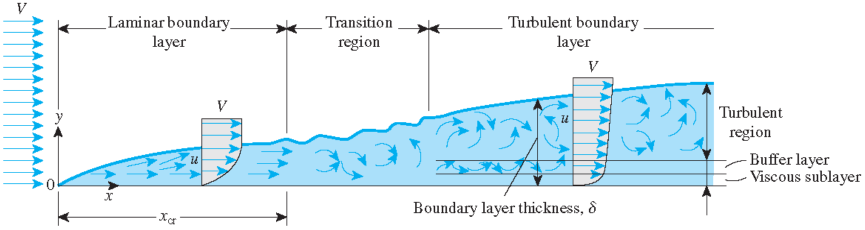

FIGURE 6–14
The development of the boundary layer for flow over a flat plate, and the different flow regimes.
Courtesy of University of Delaware.

Thus we replace

$$\begin{aligned} \partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\ \partial_t u' + \left(\overline{u} + u'\right) \cdot \nabla u' + \nabla p' &= \nu \Delta u' + f' \end{aligned}$$

by the model

$$\begin{aligned} \partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\ \partial_t u' + \nabla p' &= \nu \Delta u' - \frac{1}{\epsilon} u' + \frac{1}{\epsilon} \sum_k \sigma_k \partial_t W^k \end{aligned}$$

where both equations are considered in the full domain D but the second one is mostly active near the boundary thanks to the fact that the vector fields σ_k have small support near the boundary.

Closed model for large scales

Let us look only at the equation of large scales

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \nu \Delta \overline{u} + \overline{f} - u' \cdot \nabla \overline{u}.$$

If we take the limit $\epsilon \to 0$ and argue as in the linear case of temperature diffusion, we get the equation

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = (\nu \Delta + \mathcal{L}) \, \overline{u} + \overline{f} - \sum_{k \in K} (\sigma_k \cdot \nabla \overline{u}) \, \partial_t W^k.$$

This is a *closed* model of large scales, influenced by turbulent small scales.

Is it useful and realistic?

This difficult question is under investigation. Let us only mention one positive fact. Consider the associated deterministic equation

$$\begin{aligned} \partial_t U + U \cdot \nabla U + \nabla P &= (\nu\Delta + \mathcal{L})\, U + \overline{f} \\ \mathrm{div}\, U &= 0, \qquad U|_{\partial D} = 0, \qquad u'(0) = \overline{u}_0 \end{aligned}$$

- This equation has, for suitalbe $\mathcal{L}$, stronger dissipativity properties.
- In $d = 2$ we can prove that $\overline{u}$ is close to U for suitable noise. This is the observed phenomenon of *eddy viscosity*: turbulence improves the viscous properties.

The 3D Navier-Stokes equations with just transport

Preliminary: define the *vorticity* as

$$\omega = \operatorname{curl} u$$

$$\omega \overset{d=2}{=} \partial_2 u_1 - \partial_1 u_2.$$

It satisfies the equation

$$\partial_t \omega + \underbrace{u \cdot \nabla \omega}_{\text{transport}} = \nu \Delta \omega + \underbrace{\omega \cdot \nabla u}_{\text{stretching}} + \operatorname{curl} f.$$

$$\partial_t \omega + u \cdot \nabla \omega \overset{d=2}{=} \nu \Delta \omega + \operatorname{curl} f$$

The 2D case

Now, apply stochastic model reduction as above to the vorticity:

$$\partial_t \omega + u \cdot \nabla \omega \overset{d=2}{=} \nu \Delta \omega - u' \cdot \nabla \omega + \mathrm{curl}\, f$$

$$\downarrow$$

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} \overset{d=2}{=} \nu \Delta \overline{\omega} - \sum_{k \in K} \sigma_k \cdot \nabla \overline{\omega} \circ \partial_t W^k + \overline{\mathrm{curl}\, f}.$$

This is an excellent equation, similar to the one of temperature diffusion and transport. In particular, one can prove (suitable noise) that $\overline{\omega}$ is close to the deterministic solution of

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=2}{=} (\nu \Delta + \mathcal{L})\, \Omega + \overline{\mathrm{curl}\, f}.$$

3D case with transport and stretching

$$\partial_t \omega + \underbrace{u \cdot \nabla \omega}_{\text{transport}} - \underbrace{\omega \cdot \nabla u}_{\text{stretching}} = \nu \Delta \omega + \operatorname{curl} f.$$

$$\downarrow$$

$$\partial_t \overline{\omega} + \left(\overline{u} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{u}\right) \overset{d=3}{=} \nu \Delta \overline{\omega} + \overline{\operatorname{curl} f}$$
$$- \sum_{k \in K} \left(\sigma_k \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \sigma_k\right) \circ \partial_t W^k$$

But the link with an equation of the form

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=3}{=} \left(\nu \Delta + \mathcal{L}\right) \Omega + \Omega \cdot \nabla U + \overline{\operatorname{curl} f}$$

is not undestood until now.

3D case with only transport

On the contrary, if we investigate the model, in 3D, with just transport noise,

$$\partial_t \overline{\omega} + \left(\overline{u} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{u} \right) \overset{d=3}{=} \nu \Delta \overline{\omega} + \overline{\mathrm{curl}\, f}$$
$$- \sum_{k \in K} P \left(\sigma_k \cdot \nabla \overline{\omega} \right) \circ \partial_t W^k$$

it is possible to prove a rigorous link with

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=3}{=} \left(\nu \Delta + \mathcal{L} \right) \Omega + \Omega \cdot \nabla U + \overline{\mathrm{curl}\, f}$$

Notice that we have introduced the projection $P : L^2 \to H$ in this equation: in general the term $\sigma_k \cdot \nabla \overline{\omega}$ is not divergence free, while the sum of all other terms is divergence free.

3D case with only transport

- One can prove (F.F. - Dejun Luo, PTRF 2021) that the solution $\overline{\omega}$ of the stochastic Navier-Stokes equations is close (in a suitable topology) to the solution Ω of the deterministic Navier-Stokes equations with increased dissipation.
- This fact implies that *well-posedness is improved by noise*.
- [In the deterministic case, the larger is the viscosity, the longer is the time interval of existence and uniqueness of smooth solutions; this interval becomes even infinite when the sizes of the initial condition/forcing and the viscosity satisfy a certain relation.]
- This is the first known regularization by noise result for 3D Navier-Stokes equations. It leaves open the very difficult question whether the same result holds when the noise affect also the stretching term.

Summary

- In this chapter we discusstransport noise. Transport-stretching type in 3D is less understood.
- It introduces, by Wong-Zakai limit, an auxiliary elliptic operator.
- In the case of heat transport it proves the property of eddy dissipation.
- Similar ideas may be applied to the internal structure of the fluid, by a large/small scale analysis and stochastic modeling of small scales.
- In 2D it explains eddy viscosity: turbulence enhances the viscosity of the fluid itself

Summary

- In 3D, just transport noise (no stretching noise): it improves the theory of 3D Navier-Stokes equations, delaying the blow-up of smooth solutions.
- Deep research is needed to understand the case of transport-stretching noise.
- Heurisitc remark:
  - we started from additive perturbations motivated by the roughness of boundaries
  - additive noise in the small scales lead to multiplicative transport noise in the large scales
  - transport noise has a better regularizing power.
- At the end it seems that *it is the additive noise at small scales which regularizes*!