Stochastic Partial Differential Equations in Fluid Mechanics
Lecture 3: Stochastic Navier-Stokes equations

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Summary (and outline)

- In the first lecture we have discussed the origin of *noise from boundary perturbations*
- And we have started the rigorous investigation of *deterministic NS eq. with rough data*
- In the second lecture we have completed the rigorous investigation of deterministic NS eq. with rough data
- And we have introduced *stochastic NS eq., Itô formula, average energy identity*
- This identifies an open problem, namely that *additive noise introduces energy in the average*, so we should subtract it with some additional term.

(Summary and) outline

- Today we discuss *state-dependent noise* from the physical view-point
- and we prove rigorous results.
- The next two lectures will be devoted to *noise of transport type*
- discussing its physical origin from large-small scale decomposition
- and its consequences on turbulence theory.

Summary: Navier-Stokes equations with additive noise

$$
\begin{aligned}
du + (u \cdot \nabla u + \nabla p)\, dt &= (\nu \Delta u + f)\, dt + + \sum_{k \in K} \sqrt{\lambda_k} \sigma_k dW_t^k \\
\operatorname{div} u &= 0
\end{aligned}
$$

$$
\begin{aligned}
u|_{\partial D} &= 0 \\
u|_{t=0} &= u_0
\end{aligned}
$$

where W_t^k are independent Brownian motions.

Theorem

If $\mathbb{E}\left\|u_0\right\|_{L^2}^2 < \infty$ *then*

$$u \in C_{\mathcal{F}}\left([0,T];H\right) \cap L^2_{\mathcal{F}}\left(0,T;V\right)$$

and

$$\mathbb{E}\left[\left\|u\left(t\right)\right\|_{L^2}^2\right] + 2\nu\int_0^t \mathbb{E}\left\|\nabla u\left(s\right)\right\|_{L^2}^2 ds = \mathbb{E}\left[\left\|u_0\right\|_{L^2}^2\right] + t\sum_{k\in K}\lambda_k\left\|\sigma_k\right\|_{L^2}^2$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u\left(t\right)\right\|_{L^2}^2\right] &\leq \mathbb{E}\left[\left\|u_0\right\|_{L^2}^2\right] + T\sum_{k\in K}\sqrt{\lambda_k}\left\|\sigma_k\right\|_{L^2}^2 \\ &\quad + C\sum_{k\in K}\lambda_k\mathbb{E}\int_0^T\left\langle u\left(s\right),\sigma_k\right\rangle^2 ds.\end{aligned}$$

Consequences

- The solution has integrability properties in ω reflecting analogous properties assumed on the data.
- Assume $u(t)$ statistically stationary solution: $\mathbb{E}\left\|u(t)\right\|_{L^2}^2 = \mathbb{E}\left\|u_0\right\|_{L^2}^2$ and $\mathbb{E}\left\|\nabla u(s)\right\|_{L^2}^2$ is independent s. Then:

$$\epsilon := \nu \mathbb{E}\left\|\nabla u_\nu\right\|_{L^2}^2 ds = \frac{1}{2}\sum_{k\in K}\lambda_k \left\|\sigma_k\right\|_{L^2}^2 .$$

The dissipation ϵ of energy due to viscosity remains constant in the inviscid limit $\epsilon \to 0$ (it is a statement of K41 theory), if the energy injection is constant.

Open problem

- Creating vortices from nothing we introduce energy into the system.
- Therefore we should include an extra dissipation mechanism. Possible proposal:

$$du + (u \cdot \nabla u + \nabla p)\, dt = \left(\nu \Delta u - \underline{\lambda(x)\, u}\right) dt + \sum_{k \in K} \sqrt{\lambda_k} \sigma_k dW_t^k$$

The energy balance is now

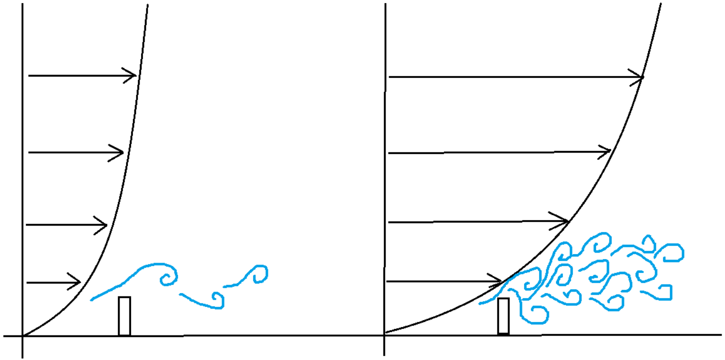
$$\begin{aligned} & \mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\nu \int_0^t \mathbb{E}\|\nabla u\|_{L^2}^2\, ds + 2\mathbb{E}\int_0^t \int_D \lambda\, |u|^2\, dxds \\ = \;& \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + t \sum_{k \in K} \lambda_k \|\sigma_k\|_{L^2}^2 . \end{aligned}$$

But we should be able to choose $\lambda(x)$ in such a way that

$$2\mathbb{E}\int_0^t \int_D \lambda(x)\, |u(s,x)|^2\, dxds \sim t \sum_{k \in K} \lambda_k \|\sigma_k\|_{L^2}^2 .$$

We do not know how to reach this target.

Example of state-dependent noise



Example of state-dependent noise

Jumps (creation of new structures) by non-homogeneous Poisson process with instantaneous rate

$$\lambda_k (u (t)) = \chi^2 \left(\frac{1}{|B (x_k, r)|} \int_{B(x_k, r)} |u (t, y)| \, dy \right)$$

where χ^2 is a nondecreasing non-negative function, equal to zero in zero and $r > 0$ is a length scale relevant to the problem. Then we introduce the cumulative rate

$$\Lambda_k (t) = \int_0^t \lambda_k (u (s)) \, ds$$

Example of state-dependent noise

Finally we modify the Poisson process N_t^k by this rate, namely we consider the process

$$N^k_{\Lambda_k(t)}.$$

The case previously considered was simply

$$\lambda_k\left(u\left(t\right)\right)=\lambda_k, \qquad \Lambda_k\left(t\right)=\lambda_k, \qquad N^k_{\lambda_k t}.$$

$$\partial_t u+u\cdot\nabla u+\nabla p=\nu\Delta u+f+F\left(u\right)+\sum_{k\in K}\frac{1}{\sqrt{2}}\sigma_k\partial_t\left(N^{k,1}_{\Lambda_k(t)}-N^{k,2}_{\Lambda_k(t)}\right)$$

Example of state-dependent noise

$$\sum_{k\in K}\frac{1}{n\sqrt{2}}\sigma_k\left(x\right)\left(N^{k,1}_{n^2\Lambda_k(t)}-N^{k,2}_{n^2\Lambda_k(t)}\right). \tag{1}$$

$$B^k_{\Lambda_k(t)}=\int_0^t\sqrt{\lambda_k\left(u\left(s\right)\right)}dW^k_s$$

(jointly in k). This result in undoubtedly advanced and not trivial even at the heuristic level but notice at least the analogy with the coefficients $\sqrt{\lambda_k}$ in the case of constant rate: when $\lambda_k\left(u\left(s\right)\right)=\lambda_k$, $\Lambda_k\left(t\right)=\lambda_k$, the previous identity reads

$$B^k_{\lambda_k t}=\int_0^t\sqrt{\lambda_k}dW^k_s=\sqrt{\lambda_k}W^k_t$$

$$\partial_t u+u\cdot\nabla u+\nabla p=\nu\Delta u+f+F\left(u\right)+\sum_{k\in K}\sigma_k\left(u\right)\partial_t W^k_t$$

by introducing the maps $\sigma_k: H\longrightarrow H$ given by

$$\sigma_k\left(u\right)\left(x\right)=\sigma_k\left(x\right)\sqrt{\lambda_k\left(u\right)}.$$

The Wong-Zakai corrector

The right corrector is:

$$F(u) = \frac{1}{2} \sum_{k \in K} D\sigma_k(u) \sigma_k(u).$$

Here by $D\sigma_k(u)$ we mean the Frechét Jacobian of $\sigma_k(u)$, which is a linear bounded operator from H to H, under suitable assumptions, and $D\sigma_k(u) \sigma_k(u)$ is the application of the linear map $D\sigma_k(u)$ to the element $\sigma_k(u)$ of H.

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ &\quad + \frac{1}{2} \sum_{k \in K} D\sigma_k(u) \sigma_k(u) + \sum_{k \in K} \sigma_k(u) \partial_t W_t^k. \end{aligned}$$

Exlpanation in one-dimension

Consider the one dimensional equation, with $\sigma(x) \geq \nu > 0$,

$$\frac{dX_t^\epsilon}{dt} = \sigma(X_t^\epsilon)\frac{dW_t^\epsilon}{dt}$$

where W_t^ϵ is an approximation of a Brownian motion W_t. It is an equation with separated variables. Then

$$\int_0^T \frac{\frac{dX_t^\epsilon}{dt}}{\sigma(X_t^\epsilon)}dt = \int_0^T \frac{dW_t^\epsilon}{dt}dt$$

$$\Phi(X_T^\epsilon) - \Phi(x_0) = W_T^\epsilon, \qquad \Phi'(x) = \frac{1}{\sigma(x)}$$

$$X_t^\epsilon = \Phi^{-1}\left(\Phi\left(x_0\right) + W_t^\epsilon\right)$$

Hence $X_\cdot^\epsilon$ converges weakly to $X_\cdot$ given by

$$X_t = \Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right).$$

From Ito formula, since

$$\begin{aligned} D\Phi^{-1}(x) &= \frac{1}{\Phi'\left(\Phi^{-1}(x)\right)} = \sigma\left(\Phi^{-1}(x)\right) \\ D^2\Phi^{-1}(x) &= \sigma'\left(\Phi^{-1}(x)\right)\sigma\left(\Phi^{-1}(x)\right) \end{aligned}$$

$$dX_t = \sigma\left(X_t\right) dW_t + \frac{1}{2}\sigma'\left(X_t\right)\sigma\left(X_t\right) dt.$$

(Link with Stratonovich integrals)

Definition

Given $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ and $f \in L^2_{\mathcal{F}}(0, T; V')$, we say that

$$u \in C_{\mathcal{F}}([0, T]; H) \cap L^2_{\mathcal{F}}(0, T; V)$$

is a weak solution if

$$\begin{aligned} & \langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ = \; & \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \int_0^t \langle f(s), \phi \rangle\, ds \\ & + \int_0^t \langle F(u(s)), \phi \rangle\, ds + \sum_{k \in K} \int_0^t \langle \sigma_k(u(s)), \phi \rangle\, dW_s^k \end{aligned}$$

for every $\phi \in D(A)$.

Theorem

For every $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ *and* $f \in L^2_{\mathcal{F}}(0, T; V')$, *there exists a unique weak solution. It satisfies*

$$
\begin{aligned}
& \mathbb{E}\left[\|u(t)\|^2_{L^2}\right] + 2\nu\mathbb{E}\int_0^t \|\nabla u(s)\|^2_{L^2}\, ds \\
= \; & \mathbb{E}\left[\|u_0\|^2_{L^2}\right] + 2\mathbb{E}\int_0^t \langle u(s), f(s) + F(u(s))\rangle\, ds \\
& + \sum_{k\in K} \mathbb{E}\int_0^t \|\sigma_k(u(s))\|^2_{L^2}\, ds.
\end{aligned}
$$

Proof of uniqueness

Let $u^{(i)}$ be two solutions. Then $w = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned} & \langle w(t), \phi\rangle - \int_0^t (b(w(s), \phi, w(s)))\, ds \\ = \; & \int_0^t \langle w(s), A\phi\rangle\, ds + \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), \phi \right\rangle ds \\ & + \sum_k \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), \phi \right\rangle dW_s^k \\ & - \int_0^t \left(b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right)\right)(s)\, ds. \end{aligned}$$

No problem to estimate the F and σ_k terms, assumed to be Lipschitz. The problem is the last term.

From Itô formula:

$$
\begin{aligned}
&\|w(t)\|_H^2 + \nu \int_0^t \|\nabla w(s)\|_H^2 \, ds \\
\leq\ & C \int_0^t \|w(s)\|_H^2 \, ds + C \int_0^t \underline{\|w(s)\|_H^2 \left(1 + \left\|u^{(2)}(s)\right\|_{\mathbb{L}^4}^2\right)} ds + M_t
\end{aligned}
$$

$$
M_t := \sum_k \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), w(s) \right\rangle dW_s^k .
$$

The difficult term coming from

$$
\int_0^t \left(b\left(u^{(2)}, w, w\right) + b\left(w, w, u^{(2)}\right)\right)(s) \, ds.
$$

Taking expected values, the cubic term "does not close".

We need now a very interesting trick that we have learned from Bjorn Schmalfuss: introduced

$$\rho_t = \exp\left(-C\int_0^t \left(1+\left\|u^{(2)}(s)\right\|_{\mathbb{L}^4}^2\right) ds\right)$$

we have, from Itô formula again,

$$\|w(t)\|_H^2 \rho_t + \nu \int_0^t \|\nabla w(s)\|_H^2 \rho_s ds \leq C\int_0^t \|w(s)\|_H^2 \rho_s ds + \widetilde{M}_t$$

$$\widetilde{M}_t := \sum_{k\in K} \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), w(s)\right\rangle \rho_s dW_s^k.$$

$$\mathbb{E}\left[\|w(t)\|_H^2 \rho_t\right] + \nu\mathbb{E}\int_0^t \|\nabla w(s)\|_H^2 \rho_s ds \leq C\int_0^t \mathbb{E}\left[\|w(s)\|_H^2 \rho_s\right] ds$$

which leads to $\mathbb{E}\left[\|w(t)\|_H^2 \rho_t\right] = 0$ by Gronwall lemma.

Existence

- Compactness criteria
- Strategies to deal with convergence of laws
- The 3D case

Deterministic Aubin-Lions type theorems

Given two Banach spaces $X \subset Y$, we say that the embedding $X \subset Y$ is compact if bounded sets of X are relatively compact in Y.

Theorem

Let $X \subset Y \subset Z$ be three Banach spaces, with continuous dense embeddings. Assume that the embedding $X \subset Y$ is compact. Let $p \in [1, \infty)$ be given. Then the embedding

$$L^p(0, T; X) \cap W^{1,1}(0, T; Z) \subset L^p(0, T; Y)$$

is compact.

Several variants exist (Simon).

Useful in the stochastic case:

Theorem

If $\alpha r > 1 - \frac{r}{p}$ $\quad (p, r \geq 1)$ *then*

$$L^p(0, T; X) \cap W^{\alpha,r}(0, T; Z) \overset{compact}{\subset} L^p(0, T; Y)$$

Here $\alpha \in (0, 1)$ and $W^{\alpha,r}(0, T; Z)$ is the space of functions $f \in L^r(0, T; Z)$ such that

$$\int_0^T \int_0^T \frac{\|f(t) - f(s)\|_Z^r}{|t - s|^{1+\alpha r}} dsdt < \infty.$$

Stochastic theory

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ compactness criteria in $L^p(\Omega)$ are not natural (although something can be done by weighted Sobolev spaces and Malliavin calculus, when $(\Omega, \mathcal{F}, \mathbb{P})$ has a special structure).
The natural approach is to consider the laws of the random objects and apply compactness arguments to these laws.
Let (X, d) be a complete metric space and $\mathcal{B}$ the Borel σ-field.
A family $\mathcal{G}$ of probability measures on $(X, \mathcal{B})$ is *tight* if for every $\epsilon > 0$ there is a compact set $K \subset X$ such that

$$\mu(K) \geq 1 - \epsilon$$

for all $\mu \in \mathcal{G}$.

Theorem (Prohorov)

A family $\mathcal{G}$ of probability measures on $(X, \mathcal{B})$ is tight if and only if it is relatively compact.

Corollary

Assume (u_N) *is a sequence of random functions from* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* $L^p(0, T; Y)$. *Assume* $\alpha r > 1 - \frac{r}{p}$ *and that for every* $\epsilon > 0$ *there are* $R_1, R_2 > 0$ *such that*

$$\begin{aligned}
\mathbb{P}\left(\|u_N\|_{L^p(0,T;X)} \geq R_1 \right) &\leq \epsilon \\
\mathbb{P}\left(\|u_N\|_{W^{\alpha,r}(0,T;Z)} \geq R_1 \right) &\leq \epsilon.
\end{aligned}$$

Then there exists a subsequence (u_{N_k}) *which converges in law, in the strong topology of* $L^p(0, T; Y)$, *to a random function* $\widetilde{u}$ *from a probability space* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ *to* $L^p(0, T; Y)$. *Moreover, if* $p, r > 1$, *we may chose* (u_{N_k}) *so that* $\widetilde{u}$ *takes also values in* $L^p(0, T; X)$ *and* $W^{\alpha,r}(0, T; Z)$. *If* u_N *are* $(\mathcal{F}_t)$*-progressively measurable, there exists* $\left(\widetilde{\mathcal{F}}_t\right)$ *such that* $\widetilde{u}$ *is* $\left(\widetilde{\mathcal{F}}_t\right)$*-progressively measurable.*

Sufficient conditions:

$$\begin{aligned} \mathbb{E}\left[\|u_N\|_{L^p(0,T;X)}\right] &\leq C \\ \mathbb{E}\left[\|u_N\|_{W^{\alpha,r}(0,T;Z)}\right] &\leq C. \end{aligned}$$

Indeed, by Markov inequality,

$$\mathbb{P}\left(\|u_N\|_{L^p(0,T;X)} \geq R_1\right) \leq \frac{C}{R_1}$$

and similarly for the second inequality, hence given $\epsilon > 0$ we can find $R_1, R_2 > 0$ with the required properties.

Galerkin approximations: 2D and 3D case

Until now we have described only the 2D case. Now it is convenient to unify something with the 3D case.
A main difference is that, after the common inequality

$$b(u, v, w) \leq \|v\|_V \|u\|_{\mathbb{L}^4} \|w\|_{\mathbb{L}^4}$$

iwe have different Sobolev embeddings::

$$\|f\|_{L^4} \overset{d=2}{\leq} \|f\|_{W^{\frac{1}{2},2}} \leq \|f\|_{L^2}^{1/2} \|f\|_{W^{1,2}}^{1/2}$$

$$\|f\|_{L^4} \overset{d=3}{\leq} \|f\|_{W^{\frac{3}{4},2}} \leq \|f\|_{L^2}^{1/4} \|f\|_{W^{1,2}}^{3/4}.$$

Notations in 2D and 3D

- (e_i) complete orthonormal system in H made of eigenvectors of A, with eigenvalues $(-\lambda_i)$,
- H_n and π_n are consequently defined
- bilinear operator $B_n : H_n \times H_n \to H_n$ definend as

$$B_n(u, v) = \pi_n P(u \cdot \nabla v)$$

- equation

$$du_n + B_n(u_n, u_n)\, dt = (Au_n + f_n + F_n(u_n))\, dt + \sum_k \sigma_k^n(u_n)\, dW_t^k$$

  where $f_n = \pi_n f$, $F_n(u) = \pi_n F(u)$, $\sigma_k^n(u_n) = \pi_n \sigma_k(u_n)$; with initial condition $u_0^n = \pi_n u_0$
- It is easy to check that

$$\langle B_n(u_n, u_n), u_n \rangle = 0.$$

Common estimates in 2D and 3D

From

$$\|u_n(t)\|_H^2 + 2\nu \int_0^t \|\nabla u_n(s)\|_H^2 \, ds = 2\int_0^t \langle f_n(s) + F(u_n(s)), u_n(s)\rangle \, ds + \sum_{k\in K} \int_0^t \|\sigma_k^n(u_n(s))\|_{L^2}^2 \, ds + M_t^n$$

we easily deduce

$$\mathbb{E}\int_0^T \|u_n(s)\|_V^2 \, ds \leq C$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^2\right] \leq C.$$

With more work (just technical, see the notes), assuming

$$\mathbb{E}\int_0^t \|f(s)\|_{V'}^r \, ds < \infty$$

for some $r > 2$, we prove

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^r\right] \leq C.$$

With more work (just technical, see the notes), we can prove

$$\mathbb{E}\int_0^T\int_0^T \frac{\left\|u_n(t)-u_n(s)\right\|_{V'}^r}{\left|t-s\right|^{1+\alpha r}}dsdt \leq C$$

if

$$1-\frac{r}{2}<\alpha r<\frac{r}{2}.$$

Given $r>1$ there exists $\alpha\in\left(0,\frac{1}{2}\right)$ with such property.

Thus we have:

Theorem

There exist (α, r) with $\alpha r > 1 - \frac{r}{2}$ and $C > 0$ such that

$$\mathbb{E}\left[\|u_n\|_{W^{\alpha,r}(0,T;V')}\right] \leq C.$$

Form the previous results:

Corollary

The family of laws of u_n is tight in $L^2(0, T; H)$. Hence there exist subsequences which converge in law.

Passage to the limit

- We have seen above that the estimates on (u_n) are similar in the 2D and 3D case.
- The problem is uniqueness: in the 2D case we have proved it. In the 3D case it is an open problem (see below).
- When we have uniqueness, we can update the convergence in law to convergence in probability (see below).
- Without uniqueness, we can pass to the limit "in law" and find solutions on an auxiliary probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$.