Stochastic Partial Differential Equations in Fluid Mechanics
Lecture 2: The Navier-Stokes equations with rough force

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The Navier-Stokes equations

Assume D is a regular bounded connected open domain. Consider the equations with *non-differentiable* forcing term W:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f + \partial_t W \\ \operatorname{div} u &= 0 \end{aligned}$$

supplemented by boundary and initial condition

$$\begin{aligned} u\big|_{\partial D} &= 0 \\ u\big|_{t=0} &= u_0 . \end{aligned}$$

Functional analysis

$$\begin{aligned} H &= \left\{ v \in L^2\left(D;\mathbb{R}^2\right) : \operatorname{div} v = 0, v \cdot n|_{\partial D} = 0 \right\} \\ V &= \left\{ v \in H^1\left(D;\mathbb{R}^2\right) : \operatorname{div} v = 0, v|_{\partial D} = 0 \right\} \\ D(A) &= \left\{ v \in H^2\left(D;\mathbb{R}^2\right) : \operatorname{div} v = 0, v|_{\partial D} = 0 \right\} \\ D(A) &\subset V \subset H \cong H' \subset V' \end{aligned}$$

$$\begin{aligned} P &: L^2\left(D;\mathbb{R}^2\right) \to H \text{ projection} \\ w &= v + \nabla q \qquad w \in L^2\left(D;\mathbb{R}^2\right),\ v = Pw \in H,\ \langle v, \nabla q \rangle = 0 \end{aligned}$$

$$A : D(A) \to H, \qquad Av = \nu P \Delta v$$

$$\mathbb{L}^4 = L^4\left(D,\mathbb{R}^2\right) \cap H$$

Functional analysis

$$\begin{aligned} b &: \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R} \\ b(u, v, w) &= \sum_{i,j=1}^{2} \int_D u_i(x)\, \partial_i v_j(x)\, w_j(x)\, dx = \int_D (u \cdot \nabla v) \cdot w dx \end{aligned}$$

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to V'$$

$$\begin{aligned} \langle B(u, v), \phi \rangle &= -b(u, \phi, v) = -\int_D (u \cdot \nabla \phi) \cdot v dx \\ B(u, v) &= P(u \cdot \nabla v) \end{aligned}$$

Definition

Given $u_0 \in H$, $f \in L^2(0,T;V')$ and $W \in L^\infty(0,T;D(A))$, we say that

$$\begin{aligned} u \quad \in \quad & C([0,T];H) \cap L^\infty\left(0,T;\mathbb{L}^4\right) \\ & + C([0,T];H) \cap L^2(0,T;V) \end{aligned}$$

is a weak solution of the NS equations if

$$u - z \in C([0,T];H) \cap L^2(0,T;V)$$

where z is defined above with $z_0 = 0$ and

$$\begin{aligned} & \langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ = \quad & \langle u_0, \phi \rangle + \int_0^t \langle u, A\phi \rangle\, ds + \int_0^t \langle f, \phi \rangle\, ds + \langle W(t), \phi \rangle - \langle W(0), \phi \rangle \end{aligned}$$

for every $\phi \in D(A)$.

Main result and strategy

Theorem

Assume $u_0 \in H$, $f \in L^2(0, T; V')$ and $W \in L^\infty(0, T; D(A))$. Then the Navier-Stokes equations have a unique weak solution, given by the sum of the solution.

The solution of the Navier-Stokes equations is given by the sum of the solution of Stokes problem and the solution of the auxiliary problem.

Stokes problem

Definition

Given $z_0 \in H$ and $W \in L^\infty(0, T; D(A))$, we say that z is a weak solution of Stokes problem if

$$z \in L^\infty(0, T; H)$$

and

$$\langle z(t), \phi \rangle = \langle z_0, \phi \rangle + \int_0^t \langle z(s), A\phi \rangle \, ds + \langle W(t), \phi \rangle - \langle W(0), \phi \rangle$$

for every $\phi \in D(A)$.

Theorem

If $z_0 \in H$ and $W \in L^\infty(0, T; D(A))$, then there exists one and only one weak solution of Stokes problem; it is given by

$$z(t) = e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t e^{(t-s)A} A W(s) \, ds.$$

Idea of proof

Step 1 (uniqueness and explicit formula). Let z be a solution. Let

$$\phi \in C^1\left([0,T];H\right) \cap C\left([0,T];D(A)\right)$$

be given. Then

$$\begin{aligned} & \langle z(t), \phi(t) \rangle - \langle z_0, \phi(0) \rangle \\ = \; & \int_0^t \langle z(s), \partial_s \phi(s) \rangle \, ds + \int_0^t \langle z(s), A\phi(s) \rangle \, ds \\ & + \langle W(t), \phi(t) \rangle - \langle W(0), \phi(0) \rangle - \int_0^t \langle W(s), \partial_s \phi(s) \rangle \, ds. \end{aligned}$$

Take the function

$$\phi_t(s) := e^{(t-s)A}\psi$$

with $\psi \in D\left(A^2\right)$.

$$
\begin{aligned}
& \langle z(t), \phi(t)\rangle - \langle z_0, \phi(0)\rangle \\
= \; & \int_0^t \langle z(s), \partial_s \phi(s)\rangle \, ds + \int_0^t \langle z(s), A\phi(s)\rangle \, ds \\
& + \langle W(t), \phi(t)\rangle - \langle W(0), \phi(0)\rangle - \int_0^t \langle W(s), \partial_s \phi(s)\rangle \, ds.
\end{aligned}
$$

$$
\phi_t(s) := e^{(t-s)A}\psi
$$

$$
\begin{aligned}
& \langle z(t), \psi\rangle - \left\langle z_0, e^{tA}\psi \right\rangle \\
= \; & -\int_0^t \left\langle z(s), Ae^{(t-s)A}\psi \right\rangle ds + \int_0^t \left\langle z(s), Ae^{(t-s)A}\psi \right\rangle ds \\
& + \langle W(t), \psi\rangle - \left\langle W(0), e^{tA}\psi \right\rangle + \int_0^t \left\langle W(s), Ae^{(t-s)A}\psi \right\rangle ds.
\end{aligned}
$$

$$
z(t) = e^{tA}z_0 + W(t) - e^{tA}W(0) + \int_0^t e^{(t-s)A} AW(s) \, ds.
$$

$\Rightarrow$ uniqueness and explicit formula.

Step 2 (existence).

$$z(t) = e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t e^{(t-s)A} A W(s)\, ds.$$

$$z(t) - W(t) = e^{tA} (z_0 - W(0)) + \int_0^t e^{(t-s)A} A W(s)\, ds$$

$$\frac{d}{dt}(z(t) - W(t)) = Az(t) - W(t) + AW(t)$$

$$\langle z(t), \phi \rangle = \langle z_0, \phi \rangle + \int_0^t \langle z(s), A\phi \rangle\, ds + \langle W(t), \phi \rangle - \langle W(0), \phi \rangle.$$

More general assumptions and additional regularity

$$z(t) = e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t A e^{(t-s)A} W(s)\, ds.$$

Theorem

If $z_0 \in D\left((-A)^{\frac{1}{4}+\epsilon}\right)$ *and*

$$W \in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right)$$

for some $\epsilon > 0$ *then*

$$z \in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\frac{\epsilon}{2}}\right)\right) \subset L^\infty\left(0, T; \mathbb{L}^4\right).$$

$$\begin{aligned} z_0 &\in D\left((-A)^{\frac{1}{4}+\epsilon}\right) \\ W &\in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right) \end{aligned}$$

$$z(t) = \underbrace{e^{tA} z_0 + W(t) - e^{tA} W(0)}_{\in L^\infty\left(0,T;D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right)} + \int_0^t A e^{(t-s)A} W(s)\, ds$$

$$(-A)^{\frac{1}{4}+\frac{\epsilon}{2}} \int_0^t A e^{(t-s)A} W(s)\, ds = \int_0^t \underbrace{(-A)^{1-\frac{\epsilon}{2}} e^{(t-s)A}}_{\|\cdot\|_{\mathcal{L}(H,H)} \leq C/(t-s)^{1-\epsilon/2}} \underbrace{(-A)^{\frac{1}{4}+\epsilon} W(s)}_{\in L^\infty(0,T;H)} ds.$$

Auxiliary Navier-Stokes type equations

$$v(t) = u(t) - z(t), \qquad \pi(t) = p(t) - q(t)$$

satisfy

$$\begin{aligned} \partial_t v + (v + z) \cdot \nabla (v + z) + \nabla \pi &= \nu \Delta v + f \\ \mathrm{div}\, v &= 0 \end{aligned}$$

$$\begin{aligned} v\big|_{\partial D} &= 0 \\ v\big|_{t=0} &= v_0 . \end{aligned}$$

Definition

Given $v_0 \in H$, $f \in L^2(0, T; V')$ and $z \in L^4\left(0, T; \mathbb{L}^4\right)$, we say that

$$v \in C([0, T]; H) \cap L^2(0, T; V)$$

is a weak solution of the modified NS equations if

$$\begin{aligned} & \langle v(t), \phi\rangle - \int_0^t b(v(s) + z(s), \phi, v(s) + z(s))\, ds \\ = \; & \langle v_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\, ds + \int_0^t \langle f(s), \phi\rangle\, ds \end{aligned}$$

for every $\phi \in D(A)$.

Theorem

For every $v_0 \in H$, $f \in L^2(0, T; V')$ *and* $z \in L^4(0, T; \mathbb{L}^4)$, *there exists a unique weak solution of the modified NS equations. It satisfies*

$$\begin{aligned} & \|v(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla v(s)\|_{L^2}^2 \, ds \\ = & \|v_0\|_{L^2}^2 + 2 \int_0^t \langle f(s), v(s) \rangle \, ds \\ & +2 \int_0^t (b(v, v, z) + b(z, v, v) + b(z, v, z))(s) \, ds. \end{aligned}$$

Step 1 (uniqueness). Let $v^{(i)}$ be two solutions. The function $w = v^{(1)} - v^{(2)}$ satisfies

$$\begin{aligned} & \langle w(t), \phi \rangle - \int_0^t \left(b\left(v^{(1)} + z, \phi, v^{(1)} + z\right) - b\left(v^{(2)} + z, \phi, v^{(2)} + z\right) \right. \\ = & \int_0^t \langle w(s), A\phi \rangle \, ds \end{aligned}$$

for every $\phi \in D(A)$. A simple manipulation gives us

$$\begin{aligned} & b\left(v^{(1)} + z, \phi, v^{(1)} + z\right) - b\left(v^{(2)} + z, \phi, v^{(2)} + z\right) - b(w, \phi, w) \\ = & \; b\left(v^{(2)} + z, \phi, w\right) + b\left(w, \phi, v^{(2)} + z\right) \end{aligned}$$

hence

$$\langle w(t), \phi \rangle - \int_0^t b(w, \phi, w) \, ds = \int_0^t \langle w, A\phi \rangle \, ds + \int_0^t \left\langle \widetilde{f}, \phi \right\rangle ds$$

$$\widetilde{f} = -B\left(v^{(2)} + z, w\right) - B\left(w, v^{(2)} + z\right).$$

By the deterministic Theorem,

$$\begin{aligned} &\left\| w\left(t\right)\right\|_{L^2}^2 + 2\nu \int_0^t \left\| \nabla w\left(s\right)\right\|_{L^2}^2 ds \\ = \quad & -2\int_0^t \left(b\left(v^{(2)} + z, w, w\right) - b\left(w, w, v^{(2)} + z\right)\right) ds. \end{aligned}$$

From a technical Lemma:

$$\begin{aligned} \left| b\left(v^{(2)} + z, w, w\right)\right| \quad &\leq \quad \left| b\left(v^{(2)}, w, w\right)\right| + \left| b\left(z, w, w\right)\right| \\ &\leq \quad 4\epsilon \left\| w\right\|_V^2 + \frac{C^2}{\epsilon^3} \left\| w\right\|_H^2 \left(\left\| v^{(2)}\right\|_{L^4}^4 + \left\| z\right\|_{L^4}^4 \right). \end{aligned}$$

Choose $4\epsilon = \nu$.

We get

$$\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla w(s)\|_{L^2}^2 \, ds$$
$$= \quad C \int_0^t \|w(s)\|_H^2 \left(1 + \left\|v^{(2)}(s)\right\|_{L^4}^4 + \|z(s)\|_{L^4}^4\right) ds.$$

We conclude $w = 0$ by Gronwall lemma, using the assumption

$$z \in L^4\left(0, T; \mathbb{L}^4\right)$$

and the property

$$v^{(2)} \in L^4\left(0, T; \mathbb{L}^4\right)$$

coming from the regularity

$$v^{(2)} \in C([0, T]; H) \cap L^2(0, T; V).$$

Step 2 (existence). Define the sequence (v^n) by setting $v^0 = 0$ and for every $n \geq 0$, given $v^n \in C([0, T]; H) \cap L^2(0, T; V)$, let v^{n+1} be the solution of the classical NS equations with initial condition v_0 and with

$$f + B(v^n, z) + B(z, v^n) + B(z, z)$$

in place of f. In particular

$$\begin{aligned} &\left\langle v^{n+1}(t), \phi \right\rangle - \int_0^t b\left(v^{n+1}(s), \phi, v^{n+1}(s)\right) ds \\ = \quad &\langle v_0, \phi \rangle + \int_0^t \left\langle v^{n+1}(s), A\phi \right\rangle ds + \int_0^t \langle f(s), \phi \rangle ds \\ &- \int_0^t \langle (B(v^n, z) + B(z, v^n) + B(z, z))(s), \phi \rangle ds \end{aligned}$$

for every $\phi \in D(A)$.

From the deterministic theorem:

$$\begin{aligned} & \left\| v^{n+1}(t) \right\|_{L^2}^2 + 2\nu \int_0^t \left\| \nabla v^{n+1}(s) \right\|_{L^2}^2 ds \\ = & \left\| v_0 \right\|_{L^2}^2 + 2 \int_0^t \left\langle f(s), v^{n+1}(s) \right\rangle ds \\ & +2 \int_0^t \left(b\left(v^n, v^{n+1}, z\right) + b\left(z, v^{n+1}, v^n\right) + b\left(z, v^{n+1}, z\right) \right)(s)\, ds. \end{aligned}$$

With technical work, for T small enough, we prove iteratively a uniform bound:

$$\sup_{t\in[0,T]} \left\| v^n(t) \right\|_H^2 \leq R, \qquad \int_0^T \left\| v^n(s) \right\|_V^2 ds \leq R$$

Using the previous bounds and similar arguments one can prove that the sequence (v^n) is Cauchy in $C([0,T];H) \cap L^2(0,T;V)$ (strong topology! Compare with compacteness criteria).
The limit v has the right regularity and is a weak solution of the modified NS equations. And satisfies the energy bound.
Remark:

- $W \mapsto z$ measurable
- $(v_0, f, z) \mapsto v$ measurable
- $u = z + v$ implies that $(u_0, f, W) \mapsto u$ measurable.