Chapter 1. The Navier-Stokes equations with rough force

Franco Flandoli, Scuola Normale Superiore
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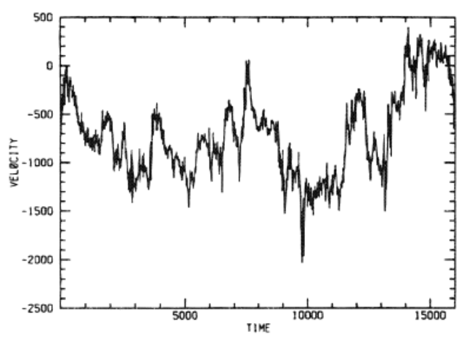
1 Introduction

1.1 Stochastic fluid dynamics

Fluid dynamics is part of continuum mechanics and its laws are deterministic. However, the observation of certain particular fluid flows reveals features which, at least on intuitive ground, look random.



Turbulent velocity signal, from Sreenivasan, Ann. Rev. Fluid Mech. 23, 1991.

The theory of deterministic dynamical systems has developed outstanding ideas to understand how a random signal may arise from a deterministic motion; its limitation for fluid dynamics is that the technical complexity of such theories does not match systems like the Navier-Stokes equations, but only more abstract examples like maps of the interval or the torus.

The theory of statistical hydrodynamics approaches the question from a statistical viewpoint; the Navier-Stokes equations very often do not play an explicit role and the main emphasis is on the constructions of stochastic processes with quantitative properties in agreements with data, independently of the relation with the equations of motion.

Stochastic fluid dynamics, the theory described in these lectures, is somewhat in between. It is always based on classical equations of continuum mechanics, but enriched by means of random elements, like for instance an additive noise or a transport noise. Needless to say, opposite to the previous two approaches, it requires a justification: where does the noisy force or coefficients come from?

1.2 Where the noise comes from

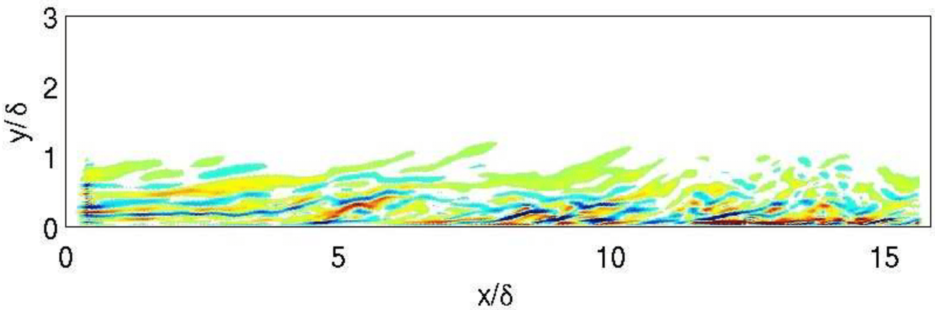
This is one of the main questions which need more accurate research. In a sense, the technical development of the mathematical theory (some elements of which are the core of these lectures) is incomparable with respect to the poor attention paid to the question about where the noise comes from.

We have four remarks, not exhaustive of all possibilities:

1. vorticity production at boundaries
2. perturbations at the interaction between different fluids or fluid/structure
3. vorticity production in shear flows
4. the dance of the vortex structures
5. how small scales affect large ones.

Our lectures will take into account 1 and 5, but just here in the introduction let us mention also 2, 3 and 4.

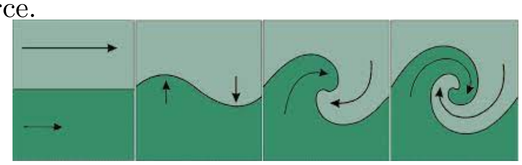
1. *Vorticity production at boundaries.* Most physical boundaries have some degree of irregularity, sometimes enormous: think of the irregularities of the hearth surface for the wind flow in the lower atmosphere layer, irregularities due to mountain chains at large scale, hills at medium scale, trees and human constructions at smaller scale. Mathematical models of fluids cannot take into account such details in a precise way, there is always some degree of simplification. But the instability of the flow at the boundary, originating vortices, is very strong, hence the frequency and intensity of creation of vortices at the boundary strongly suffers from the imprecision of the description of the true boundary.



Replacing the true details of the boundary by a random mechanism of vorticity production would increase the realism of certain flow models.

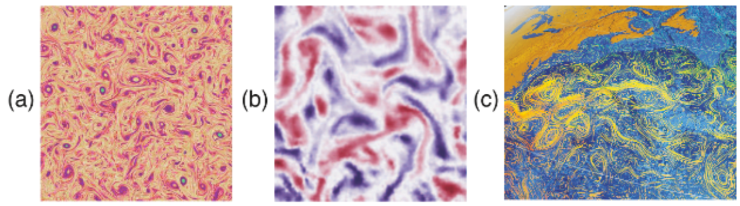
2. *Perturbations at the interaction between different fluids or fluid/structure.* This example has analogies with the previous one but it is different in the details. When we investigate the flow along the wing of an airplane we consider the boundary of the wing as the boundary of the domain of the fluid; in practice, during flight, this wing undergoes oscillations and deformations; we cannot include them in detail, unless we model carefully also all the mechanical features of the airplane. When we consider a model of the ocean, we could include the perturbations at the interface with the atmosphere which, however, are known in detail only if we include a detailed model of atmosphere.

3. *Vorticity production in shear flows.* Another very important instability in real flow is due to shear, strong difference in velocity between two nearby layers. Also in this case small local perturbations of the velocity profile lead to creation of vortices. However, the continuum mechanic modeling of a shear flow is exact, it is not approximate as in the case of a boundary. Therefore the "noise", in the sense of motion complexity, due to the emerging vortices, should already be there, in the equations; we should not introduce it by force.



Different however is when such flows are simulated. Since the resolution of any simulation is limited, it may be the case that we miss vorticity creation in shear flows due to resolution and thus it could be useful to reintroduce it by force. But since this is an issue particular of the simulation aspect, which is not the topic of these lectures, we do not insist on it.

4. *The dance of the vortex structures.*



From R. E. Ecke, J. Fluid Mech. 828, 2017.

Probably many of us have seen the movie of 2D vortex structures dancing one around the other, merging from smaller to larger ones, in a very complex manner. Emergence of stochastic features in the motion of interacting particles is a classical research topic. Therefore this is another aspect of fluid motion where a noisy signal may arise. It seems that this topic is entirely open.

5. *How small scales affect large ones.* Dividing a fluid field in a large scale component plus a small scale one is a classical procedure in fluid mechanics. Then the question is the closure, namely how to make the equation of large scales closed, with the consequent advantages for numerical simulation or for the theoretical investigation of large scale properties (we shall discuss two of them, eddy viscosity and eddy diffusion, in Chapter 4).

Usually closures are deterministic but, in case we recognize that the small scale motion has stochastic features, it may be natural to investigate closures made of stochastic equations, where the noise represents the input of small scales on large ones.

2 The deterministic Navier-Stokes equations

2.1 The Newtonian equations

The first two chapters of these notes are based on the following mathematical model, called the incompressible Navier-Stokes equations. We assume that D is a regular bounded connected open domain, but for the purpose of this introductory subsection it can be more general. In D we have a fluid described by means of its velocity $u = u(t,x)$ (a vector field) and pressure $p = p(t,x)$ (a scalar field). The equations are

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ \operatorname{div} u &= 0 \end{aligned} \tag{1}$$

supplemented by boundary and initial condition

$$\begin{aligned} u|_{\partial D} &= 0 \\ u|_{t=0} &= u_0. \end{aligned}$$

The density field is assumed to be constant and, up to a normalization, equal to 1, hence it does not explicitly appear in the equations. Constant density is the consequence of two assumptions: incompressibility, imposed by the equation $\operatorname{div} u = 0$, and the assumption that the density is constant at time zero, hence remains constant. The fluid is assumed to be viscous, namely we assume

$$\nu > 0$$

and this fact has, as a consequence, the no-slip boundary condition $u|_{\partial D} = 0$, because viscous fluids must be at rest on solid boundaries. The function f is a body force, like gravitation. The differential equation $\partial_t u + ...$ in (5) is a system, being u a vector field. The meaning of such an equation is the second Newton law: consider a very small portion of fluid, identified by a point $x(t)$, which moves in time. Recall we assume mass density equal one. The acceleration $x''(t)$ is equal to the sum of the forces. But the velocity $x'(t)$ is equal to $u(t, x(t))$, by definition of u. Hence

$$\frac{d}{dt} u(t, x(t)) = \text{forces.}$$

This reads

$$\partial_t u + u \cdot \nabla u = \text{forces}$$

along the trajectory $x(t)$, which is the first system of differential equations in (5). The forces are due to pressure, viscosity and the external ones.

We stress that the no-slip condition $u|_{\partial D} = 0$ provokes large stress near the boundary, if u is large nearby and this stress, when the viscosity is small enough, may lead to instabilities and generate vortices.

Basic is the energy balance. Assuming enough regularity to perform computations, the time derivative of the global kinetic energy is given by

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_D |u(t,x)|^2 \, dx &= \int_D u(t,x) \cdot \partial_t u(t,x) \, dx \\
&= -\int_D u \cdot (u \cdot \nabla u) \, dx - \int_D u \cdot \nabla p dx \\
&\quad +\nu \int_D u \cdot \Delta u dx + \int_D u \cdot f dx.
\end{aligned}$$

Now

$$\int_D u \cdot (u \cdot \nabla u) \, dx = \frac{1}{2}\int_D u \cdot \nabla |u|^2 \, dx = -\frac{1}{2}\int_D \operatorname{div} u \cdot |u|^2 \, dx = 0$$

(we have used also $u|_{\partial D} = 0$); similarly

$$\int_D u \cdot \nabla p dx = -\int_D p \operatorname{div} u dx = 0$$

and

$$\int_D u \cdot \Delta u dx = -\int_D |\nabla u|^2 \, dx.$$

Therefore we get

$$\frac{d}{dt}\frac{1}{2}\int_D |u(t,x)|^2\,dx + \nu\int_D |\nabla u|^2\,dx = \int_D u\cdot f dx.$$

The interpretation is that the variation of kinetic energy is given by the dissipation into heat plus the work done by the external forces. This equation is not only very informative from the physical viewpoint but represents one of the main tools in the mathematical investigation (in dimension 3, when dealing with weak solutions, it must be replaced by an inequality).

2.2 A rigorous deterministic theorem in $d = 2$

Our main aim below is giving some rigorous results about equation (5). Before we do that, it is convenient to recall a basic fact about the case of equation (1).

Assume D is a regular bounded connected open domain. Denote by $H^k\left(D,\mathbb{R}^2\right)$, $k = 1, 2, ...$the classical Sobolev spaces or vector fields and by $H_0^k\left(D,\mathbb{R}^2\right)$ the subspace of those which are zero at the boundary. Denote by H (resp. V, $D(A)$) the closure in $L^2\left(D;\mathbb{R}^2\right)$ (resp. $H^1\left(D,\mathbb{R}^2\right)$, $H^2\left(D,\mathbb{R}^2\right)$) of smooth compact support fields $v \in C_c^\infty\left(D;\mathbb{R}^2\right)$ such that $\operatorname{div} v = 0$; it turns out that it is the space of $L^2\left(D;\mathbb{R}^2\right)$-vector fields v, divergence free, such that $v\cdot n|_{\partial D} = 0$ where n is the normal to ∂D (one can prove that $v\cdot n|_{\partial D}$ is well defined, for divergence free L^2 vector fields). Denote by P the projection of $L^2\left(D;\mathbb{R}^2\right)$ on H.

Denote by V (resp. $D(A)$) the space of all $v \in H_0^1\left(D,\mathbb{R}^2\right)$ (resp. $v \in H^2\left(D,\mathbb{R}^2\right)\cap H_0^1\left(D,\mathbb{R}^2\right)$) such that $\operatorname{div} v = 0$ (they can be defined as above as the closure of smooth fields; now the Dirichlet boundary condition passes to the closure).

Define the unbounded linear operator $A : D(A) \subset H \to H$ by the identity

$$\langle Av, w\rangle = \nu\langle \Delta v, w\rangle$$

for all $v \in D(A)$ and $w \in H$, or as

$$Av = \nu P\Delta v.$$

Denote by $\mathbb{L}^4$ the space $L^4\left(D,\mathbb{R}^2\right)\cap H$, with the usual topology of $L^4\left(D,\mathbb{R}^2\right)$. Define the trilinear form $b : \mathbb{L}^4\times V\times\mathbb{L}^4 \to \mathbb{R}$ as

$$b(u,v,w) = \sum_{i,j=1}^{2}\int_D u_i(x)\,\partial_i v_j(x)\,w_j(x)\,dx = \int_D w\cdot(u\cdot\nabla v)\,dx$$

(it is well defined and continuous on $\mathbb{L}^4\times V\times\mathbb{L}^4$ by Hölder inequality). Notice that

$$V \subset \mathbb{L}^4$$

by Sobolev embedding theorem, hence b is also defined and continuous on $V \times V \times V$. Moreover, the following interpolation inequality holds true: for some constant $C > 0$

$$\|f\|_{L^4(D)}^2 \leq C \|f\|_{L^2(D)} \|f\|_{H^1(D)} \tag{2}$$

for all $f \in H^1(D)$. It follows that

$$\int_0^T \|u(t)\|_{\mathbb{L}^4}^4 \, dt \leq C \sup_{t \in [0,T]} \|u(t)\|_H^2 \int_0^T \|u(t)\|_V^2 \, dt. \tag{3}$$

This implies in particular that the integral

$$\int_0^t b(u(s), \phi, u(s)) \, ds$$

in the definition below is well defined, under the regularity of u and ϕ specified there.

Sometimes we shall also use the operator

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to V'$$

defined by the identity

$$\langle B(u,v), \phi \rangle = -b(u, \phi, v)$$

for all $\phi \in V$. It is explicitly given by

$$B(u,v) = P(u \cdot \nabla v)$$

when u, v are more regular, or by a suitable distributional interpretation of $u \cdot \nabla v$ and extension of P, which we omit since it is not essential later on; for smooth diveregence free fields, equal to zero at the boundary, we have

$$\langle B(u,v), \phi \rangle = \int_D (u \cdot \nabla v) \cdot \phi dx = -\int_D (u \cdot \nabla \phi) \cdot v dx = -b(u, \phi, v).$$

In the sequel we denote by V' the dual of V. We may identify H with H' and thus write $D(A) \subset V \subset H \subset V'$. The scalar product $\langle \cdot, \cdot \rangle$ in H "extends" to the dual pairing between V and V', which will be denoted by the same notation.

Definition 1 *Given $u_0 \in H$ and $f \in L^2(0,T;V')$, we say that*

$$u \in C([0,T];H) \cap L^2(0,T;V)$$

is a weak solution of equation (1) if

$$\begin{aligned} & \langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s)) \, ds \\ = \quad & \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle \, ds + \int_0^t \langle f(s), \phi \rangle \, ds \end{aligned}$$

for every $\phi \in D(A)$.

The previous definition is a natural reformulation of equation (1). Indeed,

$$\int_D \phi \cdot (u \cdot \nabla u)\, dx = -\int_D u \cdot (u \cdot \nabla \phi)\, dx = -b\,(u, \phi, u)$$

(using also $u|_{\partial D} = 0$) and similarly

$$\int_D \phi \cdot \Delta u dx = \int_D u \cdot \Delta \phi dx.$$

In fact we could avoid the integration by part in the first case, and a single integration by parts is sufficient in the second case, but in this way we anticipate the poor regular case investigated later on.

Theorem 2 *For every $u_0 \in H$ and $f \in L^2\,(0, T; V')$ there exists a unique weak solution of equation (1). It satisfies*

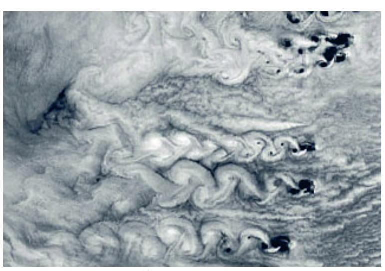
$$\|u\,(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u\,(s)\|_{L^2}^2\, ds = \|u_0\|_{L^2}^2 + 2 \int_0^t \langle u\,(s)\,, f\,(s)\rangle\, ds.$$

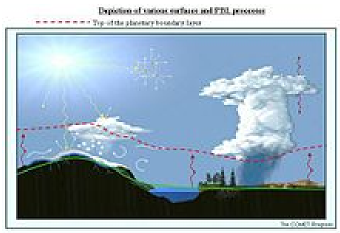
If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\omega \mapsto (u_0\,(\omega)\,, f\,(\omega))$ is a measurable map from $(\Omega, \mathcal{F})$ to $H \times L^2\,(0, T; V')$ (endowed with the Borel σ-algebra) then, called $u\,(\omega)$ the weak solution corresponding to $(u_0\,(\omega)\,, f\,(\omega))$, we have that $\omega \mapsto u\,(\omega)$ is measurable from $(\Omega, \mathcal{F})$ to $C\,([0, T]\,; H) \cap L^2\,(0, T; V)$.

We do not provide a proof but, when we give a proof for the stochastic case in Chapter 2, the reader may easily reconstruct one for this theorem.

3 Example of noise

3.1 Generation of vortices near obstacles





The precise physical description of the generation of vortices is a difficult topic in itself. Here, for the purpose of these lectures, we take a phenomenological viewpoint: emergence of vortices near obstacles is commonly observed and we content ourselves with an ad hoc inclusion of this fact into the equations. Deep research is mandatory on this issue.

Assume the velocity field at time t is $u(t,x)$. Assume that, as a consequence of an instability near the boundary, a modification occurs and in a very short time we have a field $u(t+\Delta t,x)$ which is not just equal to the smooth evolution of $u(t,x)$. We may assume that at some time t we have a jump (it is an idealization):

$$u\left(t^{+},x\right)=u\left(t^{-},x\right)+\sigma(x)$$

where $\sigma(x)$ is presumably localized in space and corresponds to a vortex structure. Continuum mechanics does not make jumps; we idealize a fast change due to an instability as a jump, to emphasize its unexpected character with respect to the unperturbed motion.

Assume that, due to several obstacles in the boundary at certain locations x_k, $k\in K$, we may observe jumps of the form

$$u\left(t^{+},x\right)=u\left(t^{-},x\right)+\sigma_k(x) \tag{4}$$

where $\sigma_k(x)$ is the perturbation around x_k. With due technical ability we may describe the possibility that, at each jump, the system makes a random choice between a very wide family of possible perturbations; here, for the sake of simplicity, we assume that K is finite.

The way to incorporate these jumps into the Navier-Stokes equations is by means of an impulsive force:

$$\partial_t u+u\cdot\nabla u+\nabla p=\nu\Delta u+\sum_{k\in K}\sum_i\delta\left(t-t_i^k\right)\sigma_k.$$

Here, for each $k\in K$, we denote by $t_1^k<t_2^k<...$ the sequence of jump times of class k. This way the fluid moves according to the free Navier-Stokes equations between two consecutive jumps times (reorder the full family $\left\{t_i^k;k\in K,i\in\mathbb{N}\right\}$ and consider two consecutive elements); and fulfils (4) at the jump times, with the correct $k\in K$.

We may assume that the jump times are random or deterministic (for the latter case, think of Karman vortices past an obstacle, as in one of the pictures above). For some of later purposes it is the same, for others it is mathematically more convenient to assume them random, thus we do so. We assume that $t_{i+1}^k-t_i^k$ has exponential distribution with mean time τ^k, $\mathbb{P}\left(t_{i+1}^k - t_i^k > s\right) = e^{s/\tau^k}$, and that all these random inter-times are independent. We may equivalently describe this by means of a family $\left\{\left(N_t^k\right)_{t\geq 0}; k \in K\right\}$ of independent standard (rate 1) Poisson processes, rescale their times as N_{t/τ^k}^k and define $t_1^k < t_2^k < ...$ as the random times when the Poisson process N_{t/τ^k}^k jumps (at time t_1^k it jumps from 0 to 1, at time t_2^k from 1 to 2 and so on). We have

$$\sum_{k\in K}\sum_i \delta\left(t - t_i^k\right)\sigma_k = \sum_{k\in K}\sigma_k \frac{dN_{t/\tau^k}^k}{dt}$$

where the time derivative of the jump process N_{t/τ^k}^k is understood in the sense of distributions.

It is then clear that we introduce the function

$$W(t,x) = \sum_{k\in K}\sigma_k(x)\, N_{t/\tau^k}^k = \sum_{k\in K}\sum_{i\in\mathbb{N}:t_i^k\leq t}\sigma_k(x)$$

and write the equation in the form

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \partial_t W. \tag{5}$$

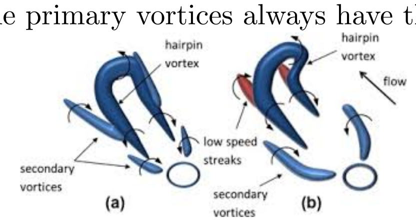
This arises the mathematical question: can we study an equation of this form when $W(t)$ is not differentiable in a classical sense?

3.1.1 The Brownian limit

In many examples the vortices appear in opposite pairs

$$\pm\sigma(x)$$

as in the wake after an obstacle (see one of the pictures above). At a boundary, usually the primary vortices always have the same sign but secondary vortices are often in pairs.



With a large degree of idealization (this issue certainly requires more investigation) let us assume that each vortex σ_k appears in pairs by means of two independent Poisson processes $N_{t/\tau^k}^{k,1}, N_{t/\tau^k}^{k,2}$ with the same rate:

$$\frac{1}{\sqrt{2}}\left(\sigma_k(x)\frac{dN_{t/\tau^k}^{k,1}}{dt} - \sigma_k(x)\frac{dN_{t/\tau^k}^{k,2}}{dt}\right).$$

The factor $\frac{1}{\sqrt{2}}$ is just to normalize and maintain the notation τ^k for the mean time between consecutive generations, now understanding the generations of $\pm\sigma_k$ as a single process. The full process $W(t,x)$ has thus the form

$$W(t,x) = \sum_{k\in K} \frac{1}{\sqrt{2}} \sigma_k(x) \left(N^{k,1}_{t/\tau^k} - N^{k,2}_{t/\tau^k} \right). \tag{6}$$

Let us parametrize by N the jump times and the vortex intensities, as:

$$W_N(t,x) = \sum_{k\in K} \frac{1}{N} \sigma_k(x) \frac{N^{k,1}_{N^2 t/\tau^k} - N^{k,2}_{N^2 t/\tau^k}}{\sqrt{2}}$$

The heuristics is that we make much more jumps but of smaller size. The precise rescaling has been chosen in order to have a non-zero finite limit. Indeed, the average of $W_N(t,x)$ is zero and the variance is equal to

$$\mathbb{E}\left[|W_N(t,x)|^2 \right] = t \sum_{k\in K} \frac{|\sigma_k(x)|^2}{\tau^k}$$

which is finite and non zero in the limit when $N \to \infty$. Let us check the previous result: since $\mathbb{E}\left[N^{k,j}_{N^2 t/\tau^k} \right] = \frac{N^2 t}{\tau^k}$, $Var\left[N^{k,j}_{N^2 t/\tau^k} \right] = \frac{N^2 t}{\tau^k}$, and $N^{k,1}_{N^2 t/\tau^k}, N^{k,2}_{N^2 t/\tau^k}$ are independent,

$$\begin{aligned} & \mathbb{E}\left[\left| \frac{1}{N} \sigma_k(x) \frac{N^{k,1}_{N^2 t/\tau^k} - N^{k,2}_{N^2 t/\tau^k}}{\sqrt{2}} \right|^2 \right] \\ = & \frac{1}{2N^2} |\sigma_k(x)|^2 \, \mathbb{E}\left[\left| N^{k,1}_{N^2 t/\tau^k} - \frac{N^2 t}{\tau^k} - N^{k,2}_{N^2 t/\tau^k} + \frac{N^2 t}{\tau^k} \right|^2 \right] \\ = & \frac{1}{2N^2} |\sigma_k(x)|^2 \, 2Var\left[N^{k,j}_{N^2 t/\tau^k} \right] = t \frac{|\sigma_k(x)|^2}{\tau^k} \end{aligned}$$

and then a similar argument applies to the sum in k.

One can thus prove that (multidimensional) Donsker invariance principle is applicable and the stochastic process $W_N(t,x)$ converges in law to

$$W(t,x) := \sum_{k\in K} \frac{1}{\sqrt{\tau^k}} \sigma_k(x) W^k_t$$

where $\left(W^k_t \right)_{t\geq 0}$ are independent Brownian motions. The Navier-Stokes equations, in the usual language of stochastic differential equations, have the form

$$du + (u \cdot \nabla u + \nabla p)\, dt = \nu \Delta u dt + \sum_{k\in K} \frac{1}{\sqrt{\tau^k}} \sigma_k dW^k_t.$$

Summarizing, we have at least two examples in mind of non-differentiable functions $W(t)$ which motivate the study of equation (5), non-classical because of the distributional time derivative: the case when $W(t)$ is a piecewise constant function, and the case when it is the trajectory of a process, linear combination of Brownian motions. Recall that, with probability one, a trajectory of Brownian motion is nowhere differentiable, not of bounded variation, not Hölder of exponent $\alpha \geq \frac{1}{2}$ on any interval, but it is locally Hölder of any exponent $\alpha < \frac{1}{2}$.

3.2 Scaling the previous example

Consider the previous system before introducing the scaling parameter N, namely equation (5) with the forcing $W(t,x)$ given by (6). Let us observe this system at a new space-time scale (if may be of interest: think to observe the hourly changes, when the vortex generation happens every few seconds). Assume $D = \mathbb{R}^2$: a cluster of islands in the ocean. Call

$$u_\lambda(t,x) := \lambda^\alpha u\left(\lambda^\beta t, \lambda x\right).$$

Then

$$\begin{aligned}
\partial_t u_\lambda(t,x) &= \lambda^{\alpha+\beta}(\partial_t u)\left(\lambda^\beta t, \lambda x\right) \\
\Delta u_\lambda(t,x) &= \lambda^{\alpha+2}(\Delta u)\left(\lambda^\beta t, \lambda x\right) \\
u_\lambda(t,x)\cdot\nabla u_\lambda(t,x) &= \lambda^{2\alpha+1}(u\cdot\nabla u)\left(\lambda^\beta t, \lambda x\right)
\end{aligned}$$

hence we have to choose $\beta = 2$ and $\alpha = 1$ to have the same multiplier, that is λ^3, and we get

$$\begin{aligned}
\partial_t u_\lambda + u_\lambda\cdot\nabla u_\lambda + \nabla p_\lambda &= \nu\Delta u_\lambda + \lambda^3(\partial_t W)\left(\lambda^2 t, \lambda x\right) \\
\operatorname{div} u_\lambda &= 0.
\end{aligned}$$

But

$$\lambda^3(\partial_t W)\left(\lambda^2 t, \lambda x\right) = \partial_t W_\lambda(t,x)$$

where

$$W_\lambda(t,x) := \lambda W\left(\lambda^2 t, \lambda x\right) = \sum_{k\in K}\frac{1}{\lambda\sqrt{2}}\sigma_k^\lambda(x)\left(N^{k,1}_{\lambda^2 t/\tau^k} - N^{k,2}_{\lambda^2 t/\tau^k}\right)$$

where

$$\sigma_k^\lambda(x) = \lambda^2\sigma_k(\lambda x).$$

Assume λ is large, like the parameter N of the previous section. In the rescaled unit of time, we make very many jumps, of larger size; but also much more concentrated, since $\sigma_k^\lambda(x)$ is rescaled as classical mollifiers.

Let us observe this force by a test function ϕ (just to avoid that the pointwise observation may suffer some regularity issue)

$$\langle W_\lambda(t), \phi\rangle = \sum_{k\in K} \frac{1}{\lambda\sqrt{2}} \left(N^{k,1}_{\lambda^2 t/\tau^k} - N^{k,2}_{\lambda^2 t/\tau^k} \right) \int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx.$$

We have zero mean and (as above)

$$\begin{aligned} \mathbb{E}\left[\langle W_\lambda(t), \phi\rangle^2\right] &= \sum_{k\in K} \frac{1}{2\lambda^2} 2\frac{\lambda^2 t}{\tau^k} \left(\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx \right)^2 \\ &= \sum_{k\in K} \frac{t}{\tau^k} \left(\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx \right)^2. \end{aligned}$$

We get

$$\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx \overset{y=\lambda x}{=} \int_{\mathbb{R}^2} \sigma_k(y)\,\phi\left(\frac{y}{\lambda}\right) dy \to \phi(0) \int_{\mathbb{R}^2} \sigma_k(y)\,dy.$$

So again we see that we have a finite non-zero limit.

What we may conclude? It is difficult to get a rich conclusion, because $\sigma_k^\lambda(x)$ converge to a vector valued space-distribution Ξ_k (a so-called current), the one such that

$$\Xi_k(\phi) = \phi(0) \int_{\mathbb{R}^2} \sigma_k(y)\,dy.$$

Thus the limit process is

$$W(t,x) := \sum_{k\in K} \frac{1}{\sqrt{\tau^k}} \Xi_k W_t^k$$

which is distributional in space, not only non-differentiable in time. Investigating this problem seems to be a challenging mathematical task.

There is a variant which should be mentioned: if we suspend the requirement that σ_k is localized and ask that the created structures are point vortices, then

$$\sigma_k(x) = \frac{1}{\pi} \frac{(x-x_0)^\perp}{|x-x_0|^2}$$

and $\sigma_k^\lambda(x) = \sigma_k(x)$! In this case the limit process is a vector field in space (not a distribution), but with infinite energy:

$$\int_{\mathbb{R}^2} |\sigma_k(x)|^2\,dx = +\infty.$$

4 Well posedness of the model with rough force

The approach we follow here may look strange at first sight but (although old) is quite modern in style. We could learn the proof of the deterministic case and adapt it to the stochastic one (Galerkin approximations, compactness etc.). This has been done with great success in the literature. However, a different approach which became more and more successful recently with singular SPDEs, consists in two steps: a probabilistic kernel often linear, Gaussian, followed by a nonlinear deterministic step. We do the same here: we solve the linear case, the so called Stokes equation, with ad hoc tools, then we apply Theorem 2. In this Chapter, thanks to the fact that the force is additive and not depending on the state of the system, we also solve the linear problem by means of deterministic tools, but in the next one we use probability there.

4.1 The Stokes problem

Let us consider first the Stokes problem:

$$\begin{aligned} \partial_t z + \nabla q &= \nu \Delta z + \partial_t W \\ \operatorname{div} z &= 0 \end{aligned}$$

Let us argue heuristically in order to identify the solution, then we formalize the concept of solution and the result. Thanks to the linearity of the problem, we may use semigroups to get an explicit formula:

$$z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} \partial_s W(s)\, ds.$$

Here we have denoted by e^{tA} the analytic semigroup generated by A. But at this level we still have the same problem of the meaning of $\partial_s W$. However, if we integrate by parts, we get

$$\begin{aligned} z(t) &= e^{tA} z_0 + \left[e^{(t-s)A} W(s) \right]_{s=0}^{s=t} - \int_0^t \frac{d}{ds} e^{(t-s)A} W(s)\, ds \\ &= e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t A e^{(t-s)A} W(s)\, ds \end{aligned}$$

which is an expression with only W. The problem now is that $Ae^{(t-s)A} W(s)$ should be well defined and integrable, in spite of the fact that A is an unbounded operator. The semigroup e^{tA}, being analytic, takes values in $D(A)$ for every $t > 0$ but with a singularity for $t = 0$, measured by the property

$$\left\| A e^{tA} h \right\|_H \leq \frac{C}{t} \left\| h \right\|_H .$$

The singularity $\frac{C}{t}$ is not integrable, hence we need some property of W in order to have that $Ae^{(t-s)A}W(s)$ is integrable on $[0,T]$.

We solve the previous problem in the simplest possible way by assuming that

$$W \in L^\infty(0,T;D(A)).$$

In the examples of the previous sections this is guaranteed by $\sigma_k \in D(A)$. Under this assumption we may write

$$\int_0^t Ae^{(t-s)A}W(s)\,ds = \int_0^t e^{(t-s)A}AW(s)\,ds$$

and the integral is obviously well defined. In the two remarks below we explain two other solutions under less regularity of W.

Remark 3 *If*

$$W \in L^\infty(0,T;D((-A)^\epsilon))$$

for some $\epsilon > 0$, then we can write

$$\int_0^t Ae^{(t-s)A}W(s)\,ds = -\int_0^t (-A)^{1-\epsilon} e^{(t-s)A}(-A)^\epsilon W(s)\,ds$$

and use the inequality

$$\left\|(-A)^{1-\epsilon} e^{tA}h\right\|_H \leq \frac{C}{t^{1-\epsilon}}\|h\|_H.$$

Remark 4 *If*

$$W \in C^\epsilon([0,T];H)$$

for some $\epsilon > 0$, then we can write

$$\begin{aligned}\int_0^t Ae^{(t-s)A}W(s)\,ds &= \int_0^t Ae^{(t-s)A}(W(s)-W(t))\,ds + \int_0^t Ae^{(t-s)A}W(t)\,ds \\ &= \int_0^t Ae^{(t-s)A}(W(s)-W(t))\,ds - W(t) + e^{tA}W(t)\end{aligned}$$

and now

$$\left\|Ae^{(t-s)A}(W(s)-W(t))\right\|_H \leq \frac{C}{t-s}|t-s|^\epsilon$$

which is integrable.

We can thus give the following definition and prove the following theorem. As just remarked, with some effort it can be extended to

$$g \in L^\infty(0,T;D((-A)^\epsilon)) + C^\epsilon([0,T];H)$$

for some $\epsilon > 0$.

Definition 5 *Given* $z_0 \in H$ *and* $W \in L^\infty(0,T;D(A))$, *we say that* z *is a weak solution of Stokes problem if*

$$z \in L^\infty(0,T;H)$$

and

$$\langle z(t), \phi\rangle = \langle z_0, \phi\rangle + \int_0^t \langle z(s), A\phi\rangle\, ds + \langle W(t), \phi\rangle - \langle W(0), \phi\rangle$$

for every $\phi \in D(A)$.

Theorem 6 *If* $z_0 \in H$ *and* $W \in L^\infty(0,T;D(A))$, *then there exists one and only one weak solution of Stokes problem; it is given by*

$$z(t) = e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t e^{(t-s)A} A W(s)\, ds. \tag{7}$$

Proof. Step 1 (uniqueness and explicit formula). Let z be a solution. Let

$$\phi \in C^1([0,T];H) \cap C([0,T];D(A))$$

be given. Let $0 = t_0 < ... < t_n = T$ be a partition of $[0,T]$, partition also denoted by π. Then, using the identities

$$\begin{aligned}
\langle z(t_{i+1}), \phi(t_{i+1})\rangle - \langle z(t_{i+1}), \phi(t_i)\rangle &= \int_{t_i}^{t_{i+1}} \langle z(t_{i+1}), \partial_s \phi(s)\rangle\, ds \\
\langle W(t_{i+1}), \phi(t_{i+1})\rangle - \langle W(t_{i+1}), \phi(t_i)\rangle &= \int_{t_i}^{t_{i+1}} \langle W(t_{i+1}), \partial_s \phi(s)\rangle\, ds
\end{aligned}$$

we get

$$\begin{aligned}
\langle z(t_{i+1}), \phi(t_{i+1})\rangle &= \langle z(t_i), \phi(t_i)\rangle + \int_{t_i}^{t_{i+1}} \langle z(t_{i+1}), \partial_s \phi(s)\rangle\, ds + \int_{t_i}^{t_{i+1}} \langle z(s), A\phi(t_i)\rangle\, ds \\
&\quad + \langle W(t_{i+1}), \phi(t_{i+1})\rangle - \langle W(t_i), \phi(t_i)\rangle - \int_{t_i}^{t_{i+1}} \langle W(t_{i+1}), \partial_s \phi(s)\rangle\, ds.
\end{aligned}$$

It implies

$$\begin{aligned}
\langle z(T), \phi(T)\rangle &= \langle z_0, \phi(0)\rangle + \int_0^T \left\langle z\left(s_\pi^+\right), \partial_s \phi(s)\right\rangle ds + \int_0^T \left\langle z(s), A\phi\left(s_\pi^-\right)\right\rangle ds \\
&\quad + \langle W(T), \phi(T)\rangle - \langle W(0), \phi(0)\rangle - \int_0^T \left\langle W\left(s_\pi^+\right), \partial_s \phi(s)\right\rangle ds
\end{aligned}$$

where $s_\pi^- = t_i$, $s_\pi^+ = t_{i+1}$, if $s \in [t_i, t_{i+1}]$. Taking the limit over a sequence of partitions π_N with size going to zero, we get

$$\begin{aligned}\langle z(T), \phi(T)\rangle &= \langle z_0, \phi(0)\rangle + \int_0^T \langle z(s), \partial_s \phi(s)\rangle\, ds + \int_0^T \langle z(s), A\phi(s)\rangle\, ds \\ &\quad + \langle W(T), \phi(T)\rangle - \langle W(0), \phi(0)\rangle - \int_0^T \langle W(s), \partial_s \phi(s)\rangle\, ds\end{aligned}$$

(thanks to the regularity of z, ϕ and Lebesgue dominated convergence theorem). The argument applies to every intermediate time t in place of T, hence we have

$$\begin{aligned}\langle z(t), \phi(t)\rangle &= \langle z_0, \phi(0)\rangle + \int_0^t \langle z(s), \partial_s \phi(s)\rangle\, ds + \int_0^t \langle z(s), A\phi(s)\rangle\, ds \\ &\quad + \langle W(t), \phi(t)\rangle - \langle W(0), \phi(0)\rangle - \int_0^t \langle W(s), \partial_s \phi(s)\rangle\, ds.\end{aligned}$$

For such value of t, take the function

$$\phi_t(s) := e^{(t-s)A}\psi$$

with $\psi \in D(A^2)$. This function is of class

$$\phi_t(\cdot) \in C^1([0,t]; H) \cap C([0,t]; D(A))$$

hence, from the previous identity,

$$\begin{aligned}\langle z(t), \psi\rangle &= \left\langle z_0, e^{tA}\psi\right\rangle - \int_0^t \left\langle z(s), Ae^{(t-s)A}\psi\right\rangle ds + \int_0^t \left\langle z(s), Ae^{(t-s)A}\psi\right\rangle ds \\ &\quad + \langle W(t), \psi\rangle - \left\langle W(0), e^{tA}\psi\right\rangle + \int_0^t \left\langle W(s), Ae^{(t-s)A}\psi\right\rangle ds.\end{aligned}$$

Using the fact that A is selfadjoint and $W(s) \in D(A)$ we get

$$\langle z(t), \psi\rangle = \left\langle e^{tA} z_0, \psi\right\rangle + \langle W(t), \psi\rangle - \left\langle e^{tA} W(0), \psi\right\rangle + \int_0^t \left\langle e^{(t-s)A} AW(s), \psi\right\rangle ds$$

and finally, by the arbitrarity of ψ, we find that z is given by the explicit formula (7). This also implies uniqueness.

Step 2 (existence). Formula (7) defines a function of class $L^\infty(0, T; H)$. The function $z(t) - W(t)$ is given by

$$z(t) - W(t) = e^{tA}(z_0 - W(0)) + \int_0^t e^{(t-s)A} AW(s)\, ds$$

and therefore, by classical results on analytic semigroups, it is differentiable for $t > 0$ and satisfies

$$\frac{d}{dt}\left(z\left(t\right) - W\left(t\right)\right) = Az\left(t\right) - W\left(t\right) + AW\left(t\right).$$

Then it is sufficient to integrate this identity in time, take the scalar product with $\phi \in D\left(A\right)$ and use the fact that A is selfadjoint. ■

When we have given the definition of the trilinear form b we have seen the role of the space $\mathbb{L}^4$. We need to upgrade the regularity of z in order to cope with the nonlinearity later on. Since it is sufficient for us, we restrict to $z_0 = 0$. As usual we state and prove the result under the abundant regularity $W \in L^\infty\left(0,T;D\left(A\right)\right)$, but the result is true, in this case, also when

$$g \in L^\infty\left(0,T;D\left(\left(-A\right)^{\frac{1}{4}+\epsilon}\right)\right) + C^{\frac{1}{4}+\epsilon}\left(\left[0,T\right];H\right)$$

for some $\epsilon > 0$.

Theorem 7 *Let $z_0 = 0$. If $W \in L^\infty\left(0,T;D\left(A\right)\right)$, then the weak solution of Stokes problem satisfies $z \in L^\infty\left(0,T;\mathbb{L}^4\right)$. If $\left(\Omega,\mathcal{F},\mathbb{P}\right)$ is a probability space and $\omega \mapsto W\left(\omega\right)$ is a measurable map from $\left(\Omega,\mathcal{F}\right)$ to $L^\infty\left(0,T;D\left(A\right)\right)$ (endowed with the Borel σ-algebra) then, called $z\left(\omega\right)$ the weak solution corresponding to $W\left(\omega\right)$, we have that $\omega \mapsto z\left(\omega\right)$ is measurable from $\left(\Omega,\mathcal{F}\right)$ to $C\left(\left[0,T\right];H\right) \cap L^\infty\left(0,T;\mathbb{L}^4\right)$.*

Proof. Without optimizing the argument, let us remark that $V \subset \mathbb{L}^4$ by Sobolev embedding theorem and

$$\left\|z\left(t\right)\right\|_V \le \left\|W\left(t\right) - e^{tA}W\left(0\right)\right\|_V + \int_0^t \left\|e^{(t-s)A}AW\left(s\right)\right\|_V ds.$$

Now $D\left(A\right) \subset V$, hence $\left\|W\left(t\right) - e^{tA}W\left(0\right)\right\|_V$ is bounded. And a well known inequality for analytic semigroups gives us, for some constant $C > 0$

$$\left\|e^{tA}w\right\|_V \le \frac{C}{\sqrt{t}}\left\|w\right\|_H$$

for all $w \in V$ and $t \in \left(0,T\right]$. Hence we deduce $z \in L^\infty\left(0,T;V\right) \subset L^\infty\left(0,T;\mathbb{L}^4\right)$. The measurability follows from the continuity, which is a consequence of linearity and boundedness. ■

4.2 Auxiliary Navier-Stokes type equations

Let us explain first the heuristics. Having solved the Stokes problem we introduce the auxiliary variable

$$v\left(t\right) = u\left(t\right) - z\left(t\right)$$

which satisfies

$$\begin{aligned}\partial_t v + (v+z)\cdot\nabla(v+z) + \nabla(p-q) &= \nu\Delta v\\ \operatorname{div} v &= 0.\end{aligned}$$

This equation has the form

$$\begin{aligned}\partial_t v + v\cdot\nabla v + \nabla\pi &= \nu\Delta v - L(v,z)\\ \operatorname{div} v &= 0\end{aligned}$$

with the affine function

$$L(v,z) = v\cdot\nabla z + z\cdot\nabla v + z\cdot\nabla z.$$

Therefore the Navier-Stokes structure is preserved, for the variable v, up to a remainder which is affine. It is then not surprising that the auxiliary equation for v is solvable similarly to the classical Navier-Stokes equations. The strategy then is solving the auxiliary equation and then deducing the solution of the Navier-Stokes equations with rough force.

To avoid a confusion with the heuristics above, let us formulate the problem from scratch. Consider the modified Navier-Stokes equation

$$\begin{aligned}\partial_t v + (v+z)\cdot\nabla(v+z) + \nabla\pi &= \nu\Delta v + f\\ \operatorname{div} v &= 0\end{aligned}\tag{8}$$

with

$$\begin{aligned}v|_{\partial D} &= 0\\ v|_{t=0} &= v_0.\end{aligned}$$

Definition 8 *Given $v_0 \in H$, $f \in L^2(0,T;V')$ and $z \in L^4\left(0,T;\mathbb{L}^4\right)$, we say that*

$$v \in C([0,T];H) \cap L^2(0,T;V)$$

is a weak solution of equation (8) if

$$\begin{aligned}&\langle v(t),\phi\rangle - \int_0^t b(v(s)+z(s),\phi,v(s)+z(s))\,ds\\ =\;&\langle v_0,\phi\rangle + \int_0^t \langle u(s),A\phi\rangle\,ds + \int_0^t \langle f(s),\phi\rangle\,ds\end{aligned}$$

for every $\phi \in D(A)$.

Theorem 9 *For every* $v_0 \in H$, $f \in L^2(0,T;V')$ *and* $z \in L^4\left(0,T;\mathbb{L}^4\right)$, *there exists a unique weak solution of equation (8). It satisfies*

$$\begin{aligned} & \|v(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla v(s)\|_{L^2}^2 \, ds \\ = \; & \|v_0\|_{L^2}^2 + 2\int_0^t \langle f(s), v(s)\rangle \, ds \\ & + 2\int_0^t \left(b(v,v,z) + b(z,v,v) + b(z,v,z)\right)(s) \, ds. \end{aligned}$$

Finally, a measurability statement completely analogous to the one of Theorem 2 holds here too.

Proof. **Step 1** (uniqueness). Let $v^{(i)}$ be two solutions. The function $w = v^{(1)} - v^{(2)}$ satisfies

$$\begin{aligned} & \langle w(t), \phi\rangle - \int_0^t \left(b\left(v^{(1)}+z, \phi, v^{(1)}+z\right) - b\left(v^{(2)}+z, \phi, v^{(2)}+z\right)\right) ds \\ = \; & \int_0^t \langle w(s), A\phi\rangle \, ds \end{aligned}$$

for every $\phi \in D(A)$. A simple manipulation gives us

$$\begin{aligned} & b\left(v^{(1)}+z, \phi, v^{(1)}+z\right) - b\left(v^{(2)}+z, \phi, v^{(2)}+z\right) - b(w,\phi,w) \\ = \; & b\left(v^{(2)}+z, \phi, w\right) + b\left(w, \phi, v^{(2)}+z\right) \end{aligned}$$

hence

$$\begin{aligned} & \langle w(t), \phi\rangle - \int_0^t b(w(s), \phi, w(s)) \, ds \\ = \; & \int_0^t \langle w(s), A\phi\rangle \, ds + \int_0^t \left\langle \widetilde{f}(s), \phi \right\rangle ds \end{aligned}$$

where

$$\widetilde{f} = -B\left(v^{(2)}+z, w\right) - B\left(w, v^{(2)}+z\right).$$

By Lemma 10 below, $\widetilde{f} \in L^2(0,T;V')$. Then, by Theorem 2,

$$\|w(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla w(s)\|_{L^2}^2 \, ds = -2\int_0^t \left(b\left(v^{(2)}+z, w, w\right) - b\left(w, w, v^{(2)}+z\right)\right) ds.$$

Again by Lemma 10, we have

$$\begin{aligned}\left|b\left(v^{(2)}+z,w,w\right)\right| &\leq \left|b\left(v^{(2)},w,w\right)\right|+\left|b\left(z,w,w\right)\right| \\ &\leq \epsilon\left\|w\right\|_V^2+\epsilon\left\|w\right\|_V^2+\frac{C^2}{\epsilon^3}\left\|w\right\|_H^2\left\|v^{(2)}\right\|_{L^4}^4 \\ &\quad +\epsilon\left\|w\right\|_V^2+\epsilon\left\|w\right\|_V^2+\frac{C^2}{\epsilon^3}\left\|w\right\|_H^2\left\|z\right\|_{L^4}^4 \\ &= 4\epsilon\left\|w\right\|_V^2+\frac{C^2}{\epsilon^3}\left\|w\right\|_H^2\left(\left\|v^{(2)}\right\|_{L^4}^4+\left\|z\right\|_{L^4}^4\right).\end{aligned}$$

Summarizing, with $4\epsilon=\nu$, using the fact that $\|w\|_V^2=\|\nabla w\|_{L^2}^2+\|w\|_H^2$, renaming the constant C,

$$\left\|w\left(t\right)\right\|_{L^2}^2+\nu\int_0^t\left\|\nabla w\left(s\right)\right\|_{L^2}^2 ds=C\int_0^t\left\|w\left(s\right)\right\|_H^2\left(1+\left\|v^{(2)}\left(s\right)\right\|_{L^4}^4+\left\|z\left(s\right)\right\|_{L^4}^4\right)ds.$$

We conclude $w=0$ by Gronwall lemma, using the assumption on z and inequality (3) for $v^{(2)}$.

Step 2 (existence). Define the sequence (v^n) by setting $v^0=0$ and for every $n\geq 0$, given $v^n\in C\left([0,T];H\right)\cap L^2\left(0,T;V\right)$, let v^{n+1} be the solution of equation (1) with initial condition v_0 and with

$$f+B\left(v^n,z\right)+B\left(z,v^n\right)+B\left(z,z\right)$$

in place of f. In particular

$$\begin{aligned}&\left\langle v^{n+1}\left(t\right),\phi\right\rangle-\int_0^t b\left(v^{n+1}\left(s\right),\phi,v^{n+1}\left(s\right)\right)ds \\ =\ &\left\langle v_0,\phi\right\rangle+\int_0^t\left\langle v^{n+1}\left(s\right),A\phi\right\rangle ds+\int_0^t\left\langle f\left(s\right),\phi\right\rangle ds \\ &-\int_0^t\left\langle\left(B\left(v^n,z\right)+B\left(z,v^n\right)+B\left(z,z\right)\right)\left(s\right),\phi\right\rangle ds\end{aligned}$$

for every $\phi\in D\left(A\right)$. In order to claim that this definition is well done, we notice that

$$B\left(v^n,z\right),B\left(z,v^n\right),B\left(z,z\right)\in L^2\left(0,T;V'\right)$$

by Lemma 10 below. ■

Then let us investigate the convergence of (v^n). First, let us prove a bound. From the previous identity and Theorem 2 we get

$$\begin{aligned}&\left\|v^{n+1}\left(t\right)\right\|_{L^2}^2+2\nu\int_0^t\left\|\nabla v^{n+1}\left(s\right)\right\|_{L^2}^2 ds \\ =\ &\left\|v_0\right\|_{L^2}^2+2\int_0^t\left\langle f\left(s\right),v^{n+1}\left(s\right)\right\rangle ds \\ &+2\int_0^t\left(b\left(v^n,v^{n+1},z\right)+b\left(z,v^{n+1},v^n\right)+b\left(z,v^{n+1},z\right)\right)\left(s\right)ds.\end{aligned}$$

It gives us (using Lemma 10 below)

$$\begin{aligned} & \left\|v^{n+1}(t)\right\|_{L^2}^2+\nu \int_0^t\left\|\nabla v^{n+1}(s)\right\|_{L^2}^2 ds \\ = & \left\|v_0\right\|_{L^2}^2+C \int_0^t\|f(s)\|_{V'}^2 ds+\epsilon \int_0^t\left\|v^n(s)\right\|_V^2 ds \\ & +C_\epsilon \int_0^t\left\|v^n(s)\right\|_H^2\left(1+\|z(s)\|_{L^4}^4\right) ds+C_\epsilon \int_0^t\|z(s)\|_{L^4}^4 ds . \end{aligned}$$

By using Gronwall lemma and a small constant ϵ, one can find $R>\left\|v_0\right\|_{L^2}^2$ and T small enough such that if

$$\sup_{t \in[0, T]}\left\|v^n(t)\right\|_H^2 \leq R, \qquad \int_0^T\left\|v^n(s)\right\|_V^2 ds \leq R \tag{9}$$

then the same inequalities hold for v^{n+1}.

Set $w_n=v^n-v^{n-1}$, for $n \geq 1$. From the identity above,

$$\begin{aligned} & \left\langle w_{n+1}(t), \phi\right\rangle+\int_0^t\left(b\left(v^{n+1}, \phi, v^{n+1}\right)-b\left(v^n, \phi, v^n\right)\right)(s) ds \\ = & \int_0^t\left\langle w_{n+1}(s), A \phi\right\rangle ds+\int_0^t\left\langle\left(B\left(v^n, z\right)-B\left(v^{n-1}, z\right)\right)(s), \phi\right\rangle ds \\ & -\int_0^t\left\langle\left(B\left(z, v^n\right)-B\left(z, v^{n-1}\right)\right)(s), \phi\right\rangle ds . \end{aligned}$$

Again as above, since

$$\begin{aligned} & b\left(v^{n+1}, \phi, v^{n+1}\right)-b\left(v^n, \phi, v^n\right)-b\left(w_{n+1}, \phi, w_{n+1}\right) \\ = & b\left(v^n, \phi, w_{n+1}\right)+b\left(w_{n+1}, \phi, v^n\right) \end{aligned}$$

we may rewrite it as

$$\begin{aligned} & \left\langle w_{n+1}(t), \phi\right\rangle+\int_0^t b\left(w_{n+1}(s), \phi, w_{n+1}(s)\right) ds \\ = & \int_0^t\left\langle w_{n+1}(s), A \phi\right\rangle ds+\int_0^t\left\langle\left(B\left(w_n, z\right)+B\left(z, w_n\right)\right)(s), \phi\right\rangle ds \\ & +\int_0^t\left(b\left(v^n, \phi, w_{n+1}\right)+b\left(w_{n+1}, \phi, v^n\right)\right)(s) ds \end{aligned}$$

One can check as above the applicability of Theorem 2 and get

$$\begin{aligned} & \left\| w_{n+1}(t) \right\|_{L^2}^2 + 2\nu \int_0^t \left\| \nabla w_{n+1}(s) \right\|_{L^2}^2 ds \\ = \; & 2 \int_0^t \left(b\left(w_n, w_{n+1}, z\right) + b\left(z, w_{n+1}, w_n\right)\right)(s)\, ds \\ & + 2 \int_0^t \left(b\left(v^n, w_{n+1}, w_{n+1}\right) + b\left(w_{n+1}, w_{n+1}, v^n\right)\right)(s)\, ds. \end{aligned}$$

As above we deduce

$$\left| b\left(v^n, w_{n+1}, w_{n+1}\right) + b\left(w_{n+1}, w_{n+1}, v^n\right)\right| \leq \frac{\nu}{2} \left\| w_{n+1} \right\|_V^2 + C \left\| w_{n+1} \right\|_H^2 \left\| v^n \right\|_{L^4}^4 .$$

But

$$\left| b\left(w_n, w_{n+1}, z\right) + b\left(z, w_{n+1}, w_n\right)\right| \leq \frac{\nu}{2} \left\| w_{n+1} \right\|_V^2 + \frac{\nu}{2} \left\| w_n \right\|_V^2 + C \left\| w_n \right\|_H^2 \left\| z \right\|_{L^4}^4 .$$

Hence

$$\begin{aligned} & \left\| w_{n+1}(t) \right\|_{L^2}^2 + \nu \int_0^t \left\| \nabla w_{n+1}(s) \right\|_{L^2}^2 ds \\ = \; & C \int_0^t \left\| w_{n+1}(s) \right\|_H^2 \left(1 + \left\| v^n(s) \right\|_{L^4}^4 \right) ds \\ & + C \int_0^t \left\| w_n(s) \right\|_V^2 ds + C \int_0^t \left\| w_n(s) \right\|_H^2 \left\| z(s) \right\|_{L^4}^4 ds. \end{aligned}$$

Now we work under the bounds (9) and deduce, using Gronwall lemma, for T possibly smalle than the previous one,

$$\begin{aligned} & \sup_{t \in [0,T]} \left\| w_{n+1}(t) \right\|_H^2 + \int_0^T \left\| w_{n+1}(s) \right\|_V^2 ds \\ \leq \; & \frac{1}{2} \left(\sup_{t \in [0,T]} \left\| w_{n+1}(t) \right\|_H^2 + \int_0^T \left\| w_{n+1}(s) \right\|_V^2 ds \right). \end{aligned}$$

It implies that the sequence (v^n) is Cauchy in $C([0,T]; H) \cap L^2(0,T;V)$. The limit v has the right regularity to be a weak solution and satisfies the weak formulation is, in the identity above for v^{n+1} and v^n we may prove that

$$\begin{aligned} \int_0^t b\left(v^{n+1}(s), \phi, v^{n+1}(s)\right) ds & \rightarrow \int_0^t b\left(v(s), \phi, v(s)\right) ds \\ \int_0^t b\left(v^n(s), \phi, z(s)\right) ds & \rightarrow \int_0^t b\left(v(s), \phi, z(s)\right) ds \\ \int_0^t b\left(z(s), \phi, v^n(s)\right) ds & \rightarrow \int_0^t b\left(z(s), \phi, v(s)\right) ds. \end{aligned}$$

All these convergeces can be easily proved by recalling the definition of b. Similarly we can pass to the limit in the energy identity.

Lemma 10 *If* $u, v \in L^4\left(0, T; \mathbb{L}^4\right)$ *then*

$$B(u, v) \in L^2\left(0, T; V'\right). \tag{10}$$

Moreover,

$$|b(u, v, w)| \leq \epsilon \|v\|_V^2 + \epsilon \|u\|_V^2 + \frac{C^2}{\epsilon^3} \|u\|_H^2 \|w\|_{L^4}^4 \tag{11}$$

$$|b(u, v, w)| \leq \epsilon \|v\|_V^2 + \epsilon \|w\|_V^2 + \frac{C^2}{\epsilon^3} \|w\|_H^2 \|u\|_{L^4}^4. \tag{12}$$

Proof. Indeed,

$$\begin{aligned} |\langle B(u, v), \phi\rangle| &= |b(u, \phi, v)| \leq \|\phi\|_V \|u\|_{L^4} \|v\|_{L^4} \\ \|B(u, v)\|_{V'} &\leq \|u\|_{L^4} \|v\|_{L^4} \end{aligned}$$

and thus

$$\int_0^T \|B(u(t), v(t))\|_{V'}^2 dt \leq \left(\int_0^T \|u(t)\|_{L^4}^4 dt\right)^{1/2} \left(\int_0^T \|v(t)\|_{L^4}^4 dt\right)^{1/2}.$$

Moreover,

$$|b(u, v, w)| \leq \|v\|_V \|u\|_{L^4} \|w\|_{L^4} \leq \epsilon \|v\|_V^2 + \frac{1}{\epsilon} \|u\|_{L^4}^2 \|w\|_{L^4}^2$$

hence the proof of (11) and (12) is the same. Let us prove the first one. From the interpolation inequality (2),

$$\begin{aligned} |b(u, v, w)| &\leq \epsilon \|v\|_V^2 + \frac{C}{\epsilon} \|u\|_V \|u\|_H \|w\|_{L^4}^2 \\ &\leq \epsilon \|v\|_V^2 + \epsilon \|u\|_V^2 + \frac{C^2}{\epsilon^3} \|u\|_H^2 \|w\|_{L^4}^4. \end{aligned}$$

■

4.3 Final main result on the equation with rough force

Finally, we may define the concept of solution and prove the well posedness for the Navier-Stokes equations with rough force

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta v + f + \partial_t W \\ \operatorname{div} u &= 0 \end{aligned} \tag{13}$$

with

$$\begin{aligned} u|_{\partial D} &= 0 \\ u|_{t=0} &= u_0. \end{aligned}$$

Definition 11 *Given* $u_0 \in H$*,* $f \in L^2(0,T;V')$ *and* $W \in L^\infty(0,T;D(A))$*, we say that*

$$\begin{aligned} u &\in C([0,T];H) \cap L^\infty\left(0,T;\mathbb{L}^4\right) \\ &\quad + C([0,T];H) \cap L^2(0,T;V) \end{aligned}$$

is a weak solution of equation (13) if

$$u - z \in C([0,T];H) \cap L^2(0,T;V)$$

where z *is defined above with* $z_0 = 0$ *and*

$$\begin{aligned} & \langle u(t), \phi\rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ = \; & \langle u_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\, ds + \int_0^t \langle f(s), \phi\rangle\, ds + \langle W(t), \phi\rangle - \langle W(0), \phi\rangle \end{aligned}$$

for every $\phi \in D(A)$.

Theorem 12 *Assume* $u_0 \in H$*,* $f \in L^2(0,T;V')$ *and* $W \in L^\infty(0,T;D(A))$*. Then the Navier-Stokes equation (13) has a unique weak solution, given by the sum of the solution of Stokes problem and the solution of the auxiliary problem, which satisfies the energy identity of Theorem 9. Finally, a measurability statement completely analogous to the one of Theorem 2 holds.*

Proof. Step 1 (uniqueness). Let $u^{(i)}$ be two solutions. Let $v^{(i)} = u^{(i)} - z$; they are solutions of the auxiliary problem, hence they coincide, hence also $u^{(i)}$ coincide.

Step 2 (existence). Let v be a solution of the auxiliary problem. Set $u = v + z$: then u is a solution of equation (13).

Step 3 (measurability). Again, $u(\omega)$ is given by

$$u(\omega) = v(\omega) + z(\omega)$$

hence it inherits the measurability properties of $v(\omega)$ and $z(\omega)$ given by Theorems 9 and 7, respectively. ■

5 Summary

The main open problem outlined in this Chapter is the link between the complexity of a real irregular boundary and stochastic models of fluids.

The main technique illustrated in the Chapter is the reduction of the PDE with rough input to the classical PDE, by means of the solution of Stokes problem with rough input.