Revisit the patch problems of 2D Boussinesq system and 2D INS system

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2D Incompressible Euler Equations

2D Euler equations in the vorticity form:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp}(-\Delta)^{-1}\omega, \quad \omega|_{t=0} = \omega_0. \end{cases} \tag{1}$$

- Global regularity of smooth solutions have been known since Wolibner [Wol33] and Hölder [Hol33].
- Yudovich [Yud63]: if $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$, then $\exists!$ global solution (u, ω) and the particle trajectory $X_t - \mathrm{Id} \in C^{\exp(-Ct\|\omega_0\|_{L^1 \cap L^\infty})}$, where X_t solves

$$\frac{\partial X_t(x)}{\partial t} = u(X_t(x), t), \quad X_t(x)|_{t=0} = x. \tag{2}$$

- Considering $\omega_0 = 1_{D_0}$, then

$$\omega(t) = 1_{D(t)}, \quad \text{with} \quad D(t) = X_t(D_0).$$

Vorticity patch problem: *whether the initial regularity of patch boundary persists globally in time*, e.g.,

$$\partial D_0 \in C^{k,\gamma}, k \in \mathbb{Z}^+, \gamma \in (0,1), \text{ whether } \partial D(t) \in C^{k,\gamma} \text{ for all time?} \tag{3}$$

Vorticity Patch Problem of 2D Euler

- Vorticity patch problem (3) was initiated in 1980s.
  Numerical simulations (e.g. Majda [Maj86]) once suggested the possibility of finite-time singularity (e.g. infinite length, corners or cusps).
- However, Chemin [Che88,Che91] proved the global persistence result of $C^{k,\gamma}$-boundary regularity, by using the paradifferential calculus and the striated regularity method.
- A simpler proof of the same result was obtained by Bertozzi & Constantin [BC93] applying the harmonic analysis techniques and contour dynamic approach.
- For other proof, see Serfati [Ser94].
- For the vorticity patch problem of 3D Euler equations, see Gamblin, Saint-Raymond [GSR95].

Density Patch Problem of INS system

Inhomogeneous Navier-Stokes (INS) equations

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho \partial_t u + \rho(u \cdot \nabla u) + \nabla p - \Delta u = 0, \\ \mathrm{div}\, u = 0, \quad (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{4}$$

- $u = (u_1, \cdots, u_d)$ velocity field, ρ density, p pressure. It models the incompressible fluid with variable densities.
- P.-L. Lions ([Lio96]) proposed Density patch problem: let ρ_0 be of patch structure, whether the regularity of patch boundary can be preserved?
- For the 2D INS with $\rho_0 = \eta_1 1_{D_0} + \eta_2 1_{D_0^c}$:
  - Liao, Zhang [LZ16]: $\partial D_0 \in W^{k,p}$, $k \geq 3$, $p \in (2, 4)$, for η_1, η_2 close to 1 and [LZ19] for any $\eta_1, \eta_2 > 0$, then $\partial D(t) \in W^{k,p}$.
  - Danchin, Zhang [DZ17a]: $\partial D_0 \in C^{1,\gamma}$, η_1, η_2 close to 1, then $\partial D(t) \in C^{1,\gamma}$.
  - Gancedo, García-Juárez [GGJ18]: $\partial D_0 \in C^{1,\gamma}$, $W^{2,\infty}$, $C^{2,\gamma}$ and $\eta_1, \eta_2 > 0$, then $\partial D(t) \in C^{1,\gamma}$, $W^{2,\infty}$, $C^{2,\gamma}$.
- For 2D INS with $\rho_0 = 1_{D_0}$: Danchin, Mucha [DM19] treated $\partial D_0 \in C^{1,\alpha}$ with $D_0 \subset \mathbb{T}^2$, then $\partial D(t) \in C^{1,\alpha}$.

Viscous Boussinesq System

The viscous Boussinesq system without heat diffusion

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \theta e_d, \\ \mathrm{div}\, u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \end{cases} \tag{5}$$

where $d = 2, 3$, $e_d = (0, \cdots, 0, 1)$, $u = (u_1, \cdots, u_d)$ velocity field, θ temperature, p pressure.

- (5) is widely used in modeling the convection phenomena in the ocean and atmospheric dynamics; it also plays an important role in studying Rayleigh-Bénard problem.
- Mathematically,
  - (5) contains incompressible Navier-Stokes and Euler as special cases;
  - 2D inviscid Boussinesq system is analogous to 3D axisymmetric Euler system away from axis.

Boussinesq Temperature Patch Problem

Boussinesq temperature patch problem for Boussinesq system (5):

Let $\theta_0 = 1_{D_0}$, with $D_0 \subset \mathbb{R}^d$ a simply connected bounded domain. Then

$$\theta(x,t) = 1_{D(t)} \quad \text{with} \quad D(t) = X_t(D_0).$$

whether the initial regularity of patch boundary persists globally in time?

- Danchin, Zhang [DZ17b] firstly proved the global well-posedness with $\theta_0 \in B^{2/q-1}_{q,1}$, $q \in (1,2)$, which admits $C^{1,\gamma}$-temperature patch 1_{D_0}, and then in 2D as well as in 3D under a smallness condition,
$$\partial D_0 \in C^{1,\gamma}, \quad \Longrightarrow \quad \partial D(t) \in C^{1,\gamma}, \quad \forall t < \infty.$$
- Gancedo, Garćia-Juárez [GGJ17] in 2D considered $\theta_0 = 1_{D_0}$ and proved $\partial D_0 \in C^{1,\gamma}, W^{2,\infty}, C^{2,\gamma} \Longrightarrow \partial D(t) \in C^{1,\gamma}, W^{2,\infty}, C^{2,\gamma}, \ \forall t < \infty$.
- Gancedo, Garćia-Juárez [GGJ20] in 3D considered more general temperature front initial data $\theta_0(x) = \theta^*(x)1_{D_0}$ with θ^* defined on $\overline{D_0}$, and under a critical smallness condition of data, the above global persistence results still hold.

Main Goal

- Note that the previous literature of both equations did not address the case of $\partial D_0 \in C^{k,\gamma}$, $k \geq 3$.
- **Main goal**: let $\partial D_0 \in C^{k,\gamma}$, $k \geq 3$, $\gamma \in (0,1)$, show the global $C^{k,\gamma}$-regularity propagation result of $\partial D(t)$ for 2D Boussinesq and 2D INS.

  Besides, for 2D Boussinesq, consider the **temperature patch of non-constant values**. It usually called the **temperature front**, and models an important physical scenario in geophysics, see Gill [Gil82], Majda [Maj03].

The Setting

- The setting for 2D Boussinesq.

  Assume $\theta_0(x) = \bar{\theta}_0(x)1_{D_0}(x)$, where $D_0 \subset \mathbb{R}^2$ is a bounded simply connected domain with $\partial D_0 \in C^{k,\gamma}$, $k \in \mathbb{Z}^+$, $\gamma \in (0,1)$.

- The setting for 2D INS.

  Assume $\rho_0(x) = \eta_1 1_{D_0}(x) + \eta_2 1_{D_0^c}(x)$, where $\eta_1, \eta_2 > 0$ (very close), $D_0 \subset \mathbb{R}^2$ is a bounded simply connected domain with $\partial D_0 \in C^{k,\gamma}$, $k \geq 3$, $\gamma \in (0,1)$.

- For both equations, consider the level-set characterization of D_0: $\exists \varphi_0 \in C^{k,\gamma}$ s.t.

$$\partial D_0 = \{x \in \mathbb{R}^2 : \varphi_0(x) = 0\},\ D_0 = \{x \in \mathbb{R}^2 : \varphi_0(x) > 0\},\ \nabla\varphi_0 \neq 0 \text{ on } \partial D_0.$$

  Then ∂D_0 can be parameterized as

$$z_0 : \mathbb{S}^1 \mapsto \partial D_0 \quad \text{with} \ \ \partial_\alpha z_0(\alpha) = \nabla^\perp \varphi_0(z_0(\alpha)) =: W_0(z_0(\alpha)),$$

  with $\nabla^\perp = (-\partial_2, \partial_1)^T$.

Main Result: 2D Boussinesq

Theorem 1 (Dongho Chae, Qianyun Miao, L. Xue, ArXiv:2110.06442v2)

Let $\theta_0(x) = \bar{\theta}_0(x) 1_{D_0}(x)$ be temperature front data with $\bar{\theta}_0 \in L^\infty(\overline{D_0})$ and $\partial D_0 \in C^{1,\gamma}(\mathbb{R}^2)$. Let $u_0 \in H^1(\mathbb{R}^2)$ be a divergence-free vector field. Then, there exists a unique global solution (θ, u) to the 2D Boussinesq system (5) such that for any $T > 0$,

$$u \in C(0,T;H^1) \cap L^2(0,T;H^2) \cap L^1(0,T;C^{1,\gamma}), \quad \forall \gamma \in (0,1),$$

$$\theta(x,t) = \bar{\theta}_0(X_t^{-1}(x)) 1_{D(t)}(x), \ \ with \ \ \partial D(t) = X_t(\partial D_0) \in L^\infty(0,T;C^{1,\gamma}),$$

where X_t is the particle-trajectory and X_t^{-1} is its inverse.

(1) *If additionally, $\partial D_0 \in W^{2,\infty}$, $\bar{\theta}_0 \in C^\mu(\overline{D_0})$, $\mu \in (0,1)$, and $u_0 \in H^1 \cap W^{1,p}$ with some $p > 2$, we get*
$$\partial D(t) \in L^\infty(0,T;W^{2,\infty}).$$

(2) *If additionally, $\partial D_0 \in C^{k,\gamma}$, $k \geq 2$, $\gamma \in (0,1)$, $\bar{\theta}_0 \in C^{k-2,\gamma}(\overline{D_0})$, and $u_0 \in H^1 \cap W^{1,p}$, $(\partial_{W_0} u_0, \cdots, \partial_{W_0}^{k-1} u_0) \in W^{1,p}$ with $p > 2$, we obtain*
$$\partial D(t) \in L^\infty(0,T;C^{k,\gamma}).$$

Here, $\partial_{W_0} u_0 := W_0 \cdot \nabla u_0$.

Remarks on Theorem 1

Remark 1

Compared with Gancedo, Garćia-Juárez [GGJ17], we consider the temperature patch problem of non-constant values, and we offer a simpler proof of the $C^{1,\gamma}$, $W^{2,\infty}$ and $C^{2,\gamma}$ regularity persistence result.

Moreover, by applying the striated estimates method initiated by Chemin [Che88,Che91], we introduce the striated type Besov space

$$\mathcal{B}^{s,\ell}_{p,r,W}(\mathbb{R}^d) := \Big\{ f \in B^s_{p,r}(\mathbb{R}^d) \,\Big|\, \|f\|_{\mathcal{B}^{s,\ell}_{p,r,W}} := \sum_{\lambda=0}^{\ell} \|\partial^\lambda_W f\|_{B^s_{p,r}} < \infty \Big\},$$

and establish a series of striated estimates in this space, and then we prove the $C^{k,\gamma}$-regularity persistence result.

Main Result: 2D INS

First recall the following $C^{1,\gamma}$-regularity persistence result with any $\eta_1, \eta_2 > 0$.

Proposition 1 (Gancedo, Garćia-Juárez, (2017), Theorem 4.1)

Let $\gamma \in (0,1)$, $s \in (0, 1-\gamma)$, $\bar{s} \in (0, s)$. Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with $\partial D_0 \in C^{1,\gamma}(\mathbb{R}^2)$, and $\rho_0(x) = \eta_1 1_{D_0}(x) + \eta_2 1_{D_0^c}(x)$ with $\eta_1, \eta_2 > 0$. Let $u_0 \in H^{\gamma+s}(\mathbb{R}^2)(\mathbb{R}^2)$ be a divergence-free vector field.
Then, there exists a unique global solution (ρ, u) to 2D INS system (4) such that for any $T > 0$,

$$u \in C(0,T; H^{\gamma+s}(\mathbb{R}^2)) \cap L^1(0,T; C^{1+\gamma+\bar{s}}(\mathbb{R}^2)), \quad \forall \bar{s} \in (0,s), \tag{6}$$

and

$$\rho(x,t) = \eta_1 1_{D(t)}(x) + \eta_2 1_{D^c(t)}(x) \quad with \ \ D(t) = X_t(D_0) \in L^\infty(0,T; C^{1,\gamma}(\mathbb{R}^2)),$$

where $X_t(\cdot)$ is the particle-trajectory.

Main Result: 2D INS

We did not achieve the goal of showing the global $C^{k,\gamma}$-persistence result for any $\eta_1, \eta_2 > 0$. We just proved this regularity result for $|\eta_1 - \eta_2|$ small enough.

Theorem 2 (Yatao Li, L. Xue, Preprint, 2021)

Under the assumptions of Proposition 1, if additionally $\partial D_0 \in C^{k,\gamma}(\mathbb{R}^2)$, $k \geq 3$, $\gamma \in (0,1)$, and $\partial_{W_0}^{\ell} u_0 \in C^{-1,\gamma}(\mathbb{R}^2)$, $\ell = 1, \cdots, k-1$, and assuming that $|\frac{\eta_1}{\eta_2} - 1| \leq c_*$ with $c_* > 0$ a sufficiently small constant, then for any $T > 0$ and any $\gamma' \in (0,\gamma)$, we have

$$\partial D(t) = X_t(\partial D_0) \in L^\infty(0,T; C^{k,\gamma'}(\mathbb{R}^2)).$$

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$$\partial D(t) = X_t(\partial D_0) \in L^\infty(0, T; C^{k,\gamma'}(\mathbb{R}^2)).$$

By replacing $C^{k,\gamma}$ with $B^{k+\gamma}_{\infty,1}$, it will be a regularity persistence result.

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Under the assumptions of Proposition 1, if additionally $\partial D_0 \in B^{k+\gamma}_{\infty,1}(\mathbb{R}^2)$, $k \geq 3$, $\gamma \in (0,1)$, *and* $\partial_{W_0}^{\ell} u_0 \in B^{\gamma-1}_{\infty,1}(\mathbb{R}^2)$, $\ell = 1, \cdots, k-1$, *and assuming that* $|\frac{\eta_1}{\eta_2} - 1| \leq c_*$ *with* $c_* > 0$ *a sufficiently small constant, then for any* $T > 0$,

$$\partial D(t) = X_t(\partial D_0) \in L^\infty(0, T; B^{k+\gamma}_{\infty,1}(\mathbb{R}^2)).$$

Remark on Theorem 2

Remark 2

- *The key point of Gancedo et al [GGJ17] is mainly using the time weighted energy estimates to step-by-step show the refined estimate of* $D_t u = (\partial_t + u \cdot \nabla) u$, *that is,* $t^{\frac{2-s}{2}} D_t u \in L^\infty_T(H^1) \cap L^2_T(H^2)$.
- *One might expect to combine the procedure of [GGJ17] with the striated estimates method to show the* $C^{k,\gamma}$*-persistence result.*

  But it will face some difficulty even at the first step, i.e., getting $\sqrt{\rho}\partial_t \partial_W u, \nabla \partial_W u \in L^2_T(L^2)$, *where* $W = \nabla^\perp \varphi$.
- *The main reason is that we need to treat the commutator term* $[\Delta, \partial_W] u = \Delta W \cdot \nabla u + 2 \nabla W \cdot \nabla^2 u$. *We only have* $W = \nabla^\perp \varphi \in C^{1,\gamma}$, $\forall \gamma \in (0,1)$, *it seems hard to treat the term like* $\int \Delta W \cdot \nabla u (\partial_t \partial_W u) \mathrm{d}x$ *in the energy type estimate.*
- *Note that in Liao, Zhang [LZ19], due to* $\partial D_0 \in W^{k,p}$, $k \geq p$, $p \in (2,4)$, *they can show* $W \in L^\infty_T(W^{2,p})$, *and so* $\Delta W \cdot \nabla u$ *and other related terms can be controlled, and time weighted energy estimates combined with striated estimates method can lead to the* $W^{k,p}$*-persistence result for any* $\eta_1, \eta_2 > 0$.

Sketch of Proof for Theorem 1

- In order to prove $C^{1,\gamma}$-, $W^{2,\infty}$-, $C^{2,\gamma}$-regularity persistence result, noting that $D(t) = X_t(D_0)$ has the level-set expression $\varphi(t)$ solving

$$\partial_t \varphi + u \cdot \nabla \varphi = 0, \quad \varphi(0, x) = \varphi_0(x), \tag{7}$$

  one needs to prove uniform boundedness of $\varphi(t)$ in $C^{1,\gamma}$, $W^{2,\infty}$, $C^{2,\gamma}$.

- A new ingredient is the introduction of **a good unknown**[1]

$$\Gamma := \omega - \mathcal{R}_{-1}\theta,$$

  with the vorticity $\omega := \partial_1 u_2 - \partial_2 u_1$ and $\mathcal{R}_{-1} := \partial_1(-\Delta)^{-1} = \partial_1 \Lambda^{-2}$.
  Note that equation of vorticity ω reads

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \partial_1 \theta, \qquad \omega|_{t=0} = \omega_0.$$

  We see $\partial_t \omega + u \cdot \nabla \omega - \Delta \Gamma = 0$, and $\partial_t \mathcal{R}_{-1}\theta + u \cdot \nabla \mathcal{R}_{-1}\theta = -[\mathcal{R}_{-1}, u \cdot \nabla]\theta$, which leads to

$$\partial_t \Gamma + u \cdot \nabla \Gamma - \Delta \Gamma = [\mathcal{R}_{-1}, u \cdot \nabla]\theta, \qquad \Gamma|_{t=0} = \Gamma_0. \tag{8}$$

[1] Such a quantity is widely used in 2D Boussinesq with partial fractional dissipation. See e.g. Hmidi, Keraani, Rousset [HKR10,HKR11].

Sketch of Proof for Theorem 1: $C^{1,\gamma}$-result

- By commutator estimate $\|[\mathcal{R}_{-1}, u\cdot\nabla]\phi\|_{B^1_{p,\infty}} \lesssim \|\nabla u\|_{L^p}\|\phi\|_{B^0_{\infty,\infty}} + \|u\|_{L^2}\|\phi\|_{L^2}$, the quantity Γ usually has good regularity estimates, e.g. $\Gamma \in \widetilde{L}^1_T(B^2_{p,\infty})$ if $u_0 \in W^{1,p}$, $p \geq 2$.
- Thus

$$\nabla u = \nabla\nabla^\perp(-\Delta)^{-1}\omega = \nabla\nabla^\perp(-\Delta)^{-1}\Gamma + \nabla\nabla^\perp(-\Delta)^{-1}\mathcal{R}_{-1}\theta. \qquad (9)$$

- Since $\theta \in L^\infty_T(L^2 \cap L^\infty)$ and $\nabla\nabla^\perp(-\Delta)^{-1}\mathcal{R}_{-1}$ is a operator of -1-order, we can prove that ∇u belong to $L^1_T(C^\gamma)$ and $X^{\pm 1}_t \in L^\infty_T(C^{1,\gamma})$, which ensures $\varphi(t) = \varphi_0(X^{-1}_t(x)) \in L^\infty_T(C^{1,\gamma})$.

Sketch of Proof for Theorem 1: $W^{2,\infty}$-result

- In order to prove that $u \in L^1_T(W^{2,\infty})$, which implies uniform $W^{2,\infty}$-boundedness of $\varphi(t)$, from (9), we mainly need to show that

$$\nabla^2\nabla^\perp(-\Delta)^{-1}\mathcal{R}_{-1}\theta \in L^\infty_T(L^\infty), \text{ with } \theta(t) = \bar{\theta}_0(X_t^{-1}(x))1_{D(t)}.$$

The situation is analogous to that in the vorticity patch problem of 2D Euler equations, where one needs to show

$$\nabla u = \nabla\nabla^\perp(-\Delta)^{-1}\omega \in L^\infty_T(L^\infty) \text{ with } \omega = 1_{D(t)}.$$

By using the additional cancellation property of the singular integral operator with even kernel (see the geometric lemma in [BC93]), we can derive the desired uniform boundedness estimate.

Sketch of Proof for Theorem 1: $C^{2,\gamma}$-result

- To obtain global uniform $C^{2,\gamma}$-estimate of $\varphi(t)$, we consider $W = \nabla^\perp\varphi$ which solves

$$\partial_t W + u\cdot\nabla W = W\cdot\nabla u = \partial_W u, \quad W|_{t=0} = W_0.$$

- By estimating C^γ-norm of $\nabla W(t)$, it mainly needs to show

$$\partial_W \nabla u \in L^1_T(C^\gamma).$$

- We see

$$\partial_W\nabla u = \partial_W\nabla\nabla^\perp\Lambda^{-2}\omega = \partial_W\nabla\nabla^\perp\Lambda^{-2}\Gamma + \partial_W\nabla\nabla^\perp\partial_1\Lambda^{-4}\theta.$$

- **Estimation of Γ-term.** Note that

$$\partial_t(\partial_W\Gamma) + u\cdot\nabla(\partial_W\Gamma) - \Delta(\partial_W\Gamma) = -\Delta W\cdot\nabla\Gamma - 2\nabla W\cdot\nabla^2\Gamma + \partial_W([\mathcal{R}_{-1}, u\cdot\nabla]\theta).$$

By using product estimate (for u divergence free)

$$\|u\cdot\nabla\phi\|_{B^{-\epsilon}_{p,r}} \lesssim \min\Big\{\|u\|_{B^{-\epsilon}_{p,r}}\|\nabla\phi\|_{L^\infty}, \|u\|_{L^\infty}\|\nabla\phi\|_{B^{-\epsilon}_{p,r}}\Big\}, \tag{10}$$

we use smoothing effect of heat eq. to get that for $0 < \gamma' < \min\{\gamma, 1 - \frac{2}{p}\}$,

$$\|\partial_W\Gamma(t)\|_{B^{\gamma'-1}_{\infty,1}} + \|\partial_W\Gamma\|_{L^1_t(B^{\gamma'+1}_{\infty,1})}$$

$$\leq Ce^{C(1+t)^2} + C\int_0^t \Big(\|\nabla u(\tau)\|_{L^\infty}\|\partial_W\Gamma(\tau)\|_{B^{\gamma'-1}_{\infty,1}} + \|W(\tau)\|_{B^{\gamma'+1}_{\infty,1}}\|\nabla\Gamma(\tau)\|_{L^\infty}\Big)\mathrm{d}\tau.$$

Combined with the following striated estimate that for $\epsilon \in (0,1)$

$$\|\partial_W(m(D)\phi)\|_{B^{-\epsilon}_{p,r}} \leq C\|\partial_W\phi\|_{B^{-\epsilon}_{p,r}} + C\|W\|_{W^{1,\infty}}\|\phi\|_{B^{-\epsilon}_{p,r}}, \tag{11}$$

we get

$$\|\partial_W(\nabla\nabla^\perp\Lambda^{-2}\Gamma)\|_{L^1_t(C^\gamma)} \lesssim e^{C(1+t)^2} + \int_0^t \Big(\|W\|_{B^{\gamma'+1}_{\infty,1}} + \|\partial_W\Gamma\|_{B^{\gamma'-1}_{\infty,1}}\Big)\Big(\|\nabla u\|_{L^\infty} + \|\Gamma\|_{L^p\cap W^{1,\infty}}\Big)\mathrm{d}\tau.$$

• **Estimation of θ-term.** Similarly,

$$\|\partial_W(\nabla\nabla^\perp\partial_1\Lambda^{-4}\theta)\|_{L^1_t(C^\gamma)} \lesssim e^{C(1+t)^2} + \|\partial_W\theta\|_{L^1_t(B^{\gamma-1}_{\infty,\infty})} + \int_0^t \|W(\tau)\|_{B^1_{\infty,1}}\|\theta(\tau)\|_{L^2\cap L^\infty}\mathrm{d}\tau.$$

▹ Note that

$$\partial_t\partial_W\theta + u\cdot\nabla\partial_W\theta = 0, \qquad \partial_W\theta|_{t=0} = \partial_{W_0}\theta_0,$$

and then

$$\|\partial_W\theta(t)\|_{B^{\gamma-1}_{\infty,\infty}} \leq e^{C\int_0^t\|\nabla u\|_{L^\infty}\mathrm{d}\tau}\|\partial_{W_0}\theta_0\|_{B^{\gamma-1}_{\infty,\infty}} \leq \|\partial_{W_0}\theta_0\|_{B^{\gamma-1}_{\infty,\infty}} e^{C(1+t)^3}.$$

Lemma 3 (Striated Regularity of Initial Temperature Front)

Let $k \geq 2$ and $0 < \gamma < 1$. Assume $D_0 \subset \mathbb{R}^2$ is a bounded simply connected domain with ∂D_0 characterized by level-set function $\varphi_0 \in C^{k,\gamma}(\mathbb{R}^2)$, and $\theta_0(x) = \bar\theta_0(x)1_{D_0}(x)$ with $\bar\theta_0 \in C^{k-2,\gamma}(\overline{D_0})$. Let $W_0 = \nabla^\perp\varphi_0$. Then

$$\partial^{k-1}_{W_0}\theta_0(x) \in C^{-1,\gamma}(\mathbb{R}^2).$$

Sketch of Proof for Theorem 1: $C^{2,\gamma}$-result

- Therefore,

$$\|W(t)\|_{B^{\gamma+1}_{\infty,\infty}} + \|\partial_W \Gamma(t)\|_{B^{\gamma'-1}_{\infty,1}} + \|\partial_W \Gamma\|_{L^1_t(B^{\gamma'+1}_{\infty,1})} + \|\partial_W \nabla u\|_{L^1_t(B^{\gamma}_{\infty,\infty})}$$

$$\leq Ce^{C(1+t)^2} + C\int_0^t \Big(\|W\|_{B^{\gamma+1}_{\infty,\infty}} + \|\partial_W \Gamma\|_{B^{\gamma'-1}_{\infty,1}}\Big)\Big(1 + \|\nabla u(\tau)\|_{C^\gamma} + \|\Gamma(\tau)\|_{L^p\cap W^{1,\infty}}\Big)\mathrm{d}\tau.$$

Gronwall's inequality guarantees

$$\|W\|_{L^\infty_T(B^{\gamma+1}_{\infty,\infty})} + \|\partial_W \Gamma\|_{L^1_T(B^{\gamma'-1}_{\infty,1})} + \|\partial_W \Gamma\|_{L^1_T(B^{\gamma'+1}_{\infty,1})} + \|\partial_W \nabla u\|_{L^1_T(B^{\gamma}_{\infty,\infty})} \leq Ce^{C(1+T)^3}.$$

Sketch of Proof for Theorem 1: Striated Estimates

- In the proof of propagation of higher $C^{k,\gamma}$-boundary regularity, motivated by [Che91,LZ16], it suffices to show the striated estimate
$$\partial_W^{k-1} W \in L_T^\infty(C^\gamma).$$
- The method of striated estimates (or conormal estimates) initiated by Chemin [Che88,Che91] plays an important role. However, the regularity of vector field W and its striated counterpart $\partial_W^\ell W$ in [Che88, Che91] are of C^γ type with $0 < \gamma < 1$, while in our situation they all belong to $C^{1,\gamma}$.
- As a consequence, it yields substantial difference in analysis. The foremost one is the estimation of R_q given by

$$R_q(\alpha_1, \cdots, \alpha_m) := \int_{[0,1]^m} \int_{\mathbb{R}^d} \prod_{i=1}^m \alpha_i(x + f_i(\tau) 2^{-q} y) h(\tau, y) \mathrm{d}y \mathrm{d}\tau.$$

Lemma 1 (Lemma A.2, Chemin (1991))

Let $\operatorname{supp} \widehat{\alpha_i} \subset B(0, C_i 2^q)$. *Let* $\epsilon \in (0,1)$, $k\epsilon < 1$, *and* W *be a regular vector field satisfying* $\|W\|_{\epsilon,W}^{1-\epsilon,k-1} = \sum_{\ell=0}^{k-1} \|(T_{W\cdot\nabla})^\ell W\|_{C^{1-\epsilon-\ell\epsilon}} < \infty$, *then for every* $\ell \le k$,

$$\left\|(T_{W\cdot\nabla})^\ell R_q(\alpha_1, \cdots, \alpha_m)\right\|_{L^\infty} \le C \sum_{|\mu| \le \ell} 2^{q(\ell-|\mu|)\epsilon} \prod_{i=1}^m \|(T_{W\cdot\nabla})^{\mu_i} \alpha_i\|_{L^\infty},$$

where $\mu = (\mu_1, \cdots, \mu_m)$ *and* C *depends on* $\|W\|_{\epsilon,W}^{1-\epsilon,k-1}$.

Sketch of Proof for Theorem 1: Striated Estimates

- Such a factor $2^{q\varepsilon(\ell-|\mu|)}$ leads to various estimates in [Che91,LZ16] with essential ϵ-regularity loss, but in our case there will be no regularity loss.
- Denote $B^s_{p,r}(\mathbb{R}^d)$ the usual Besov space, we introduce the striated type Besov space $\mathcal{B}^{s,\ell}_{p,r,W}(\mathbb{R}^d)$ and $\widetilde{\mathcal{B}}^{s,\ell}_{p,r,W}(\mathbb{R}^d)$ with

$$\|f\|_{\mathcal{B}^{s,\ell}_{p,r,W}} := \sum_{\lambda=0}^{\ell} \|\partial^\lambda_W f\|_{B^s_{p,r}} < \infty, \quad \|f\|_{\widetilde{\mathcal{B}}^{s,\ell}_{p,r,W}} := \sum_{\lambda=0}^{\ell} \|(T_{W\cdot\nabla})^\lambda f\|_{B^s_{p,r}} < \infty. \tag{12}$$

When $p=\infty$, use abbreviations $\mathcal{B}^{s,\ell}_{r,W} := \mathcal{B}^{s,\ell}_{\infty,r,W}$, $\mathcal{B}^{s,\ell}_{W} := \mathcal{B}^{s,\ell}_{1,W} = \mathcal{B}^{s,\ell}_{\infty,1,W}$.

Lemma 4

Let $\operatorname{supp}\widehat{\alpha_i} \subset B(0, C_i 2^q)$. *Let* $k \in \mathbb{Z}^+$, $\sigma \in (0,1)$ *and W be a divergence-free vector field of* $\mathbb{R}^d$ *satisfying* $\|W\|_{\widetilde{\mathcal{B}}^{1+\sigma,k-1}_{\infty,W}} := \sum_{\lambda=0}^{k-1} \|(T_{W\cdot\nabla})^\lambda W\|_{B^{1+\sigma}_{\infty,\infty}} < \infty$. *Then for every* $p \in [1,\infty]$ *and* $\ell \le k$,[a]

$$\left\|(T_{W\cdot\nabla})^\ell R_q(\alpha_1,\cdots,\alpha_m)\right\|_{L^p} \le C \min_{1\le i\le m}\Big(\sum_{|\mu|\le\ell} \|(T_{W\cdot\nabla})^{\mu_i}\alpha_i\|_{L^p} \prod_{1\le j\le m, j\ne i} \|(T_{W\cdot\nabla})^{\mu_j}\alpha_j\|_{L^\infty}\Big),$$

with C depends on $\|W\|_{\widetilde{\mathcal{B}}^{1+\sigma,k-1}_{\infty,W}}$.

[a] A similar inequality with $p=\infty$ appeared in Pg. 446 of Chemin [Che88].

Sketch of Proof for Theorem 1: Striated Estimates

Using paradifferential calculus, we establish some refined striated estimates.

Lemma 5 (Higher-order Striated Estimates)

Let $k \in \mathbb{N}$, $\sigma \in (0,1)$, and W be a divergence-free vector field satisfying

$$\|W\|_{\mathcal{B}^{1+\sigma,k-1}_{\infty,W}} := \sum_{l=0}^{k-1} \|\partial_W^l W\|_{B^{1+\sigma}_{\infty,\infty}} < \infty.$$

Let $m(D)$ be a 0-order pseudo-differential operator with $m(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Assume $u : \mathbb{R}^d \to \mathbb{R}^d$ is smooth and divergence-free, and $\phi : \mathbb{R}^d \to \mathbb{R}$ is smooth. Then for every $\epsilon \in (0,1)$ and $(p,r) \in [1,\infty]^2$, there exists $C > 0$ depending on $\|W\|_{\mathcal{B}^{1+\sigma,k-1}_{\infty,W}}$ such that:

$$\|u \cdot \nabla\phi\|_{\mathcal{B}^{-\epsilon,k}_{p,r,W}} \leq C \min\left\{ \sum_{\mu=0}^{k} \|u\|_{\mathcal{B}^{0,\mu}_{W}} \|\nabla\phi\|_{\mathcal{B}^{-\epsilon,k-\mu}_{p,r,W}}, \sum_{\mu=0}^{k} \|u\|_{\mathcal{B}^{-\epsilon,\mu}_{p,r,W}} \|\nabla\phi\|_{\mathcal{B}^{0,k-\mu}_{W}} \right\}. \tag{13}$$

$$\|m(D)\phi\|_{\mathcal{B}^{-\epsilon,k+1}_{p,r,W}} \leq C\|\phi\|_{\mathcal{B}^{-\epsilon,k+1}_{p,r,W}} + C\Big(1 + \|W\|_{\mathcal{B}^{1,k}_{W}}\Big)\Big(\|\phi\|_{\mathcal{B}^{-\epsilon,k}_{p,r,W}} + \|\Delta_{-1} m(D)\phi\|_{L^p}\Big).$$

$$\|[m(D), u\cdot\nabla]\phi\|_{\mathcal{B}^{-\epsilon,k}_{p,r,W}} \leq C\Big(\|\nabla u\|_{\mathcal{B}^{0,k}_{W}} + \|u\|_{L^\infty}\Big)\|\phi\|_{\mathcal{B}^{-\epsilon,k}_{p,r,W}}.$$

Sketch of Proof for Theorem 1: Striated Estimates

Remark 3

These striated estimates are natural generalization of some classical product and commutator estimates in the usual Besov space with negative regularity index, which might be interesting in its own.

Sketch of Proof for Theorem 1: $C^{k,\gamma}$-result

- We use the induction method. The goal is to show $\partial_W^{k-1} W \in L^\infty_T(C^\gamma)$, or more precisely, to show

$$\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,k-2}_{\infty,W})} + \|\nabla u\|_{L^1_T(\mathcal{B}^{\gamma,k-1}_{\infty,W})} + \|\Gamma\|_{L^\infty_T(\mathcal{B}^{\gamma'-1,k-1}_{W})} + \|\Gamma\|_{L^1_T(\mathcal{B}^{\gamma'+1,k-1}_{W})} \leq H_{k-1}(T).$$

- Suppose for every $\ell \in \{1, \cdots, k-2\}$ we have

$$\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,\ell-1}_{\infty,W})} + \|\nabla u\|_{L^1_T(\mathcal{B}^{\gamma,\ell}_{\infty,W})} + \|\Gamma\|_{L^\infty_T(\mathcal{B}^{\gamma'-1,\ell}_{W})} + \|\Gamma\|_{L^1_T(\mathcal{B}^{\gamma'+1,\ell}_{W})} \leq H_\ell(T), \qquad (14)$$

we intend to show the corresponding estimates with $\ell + 1$.

- Note that in showing $C^{2,\gamma}$-persistence result, we have proved (14) with $\ell = 1$.

Sketch of Proof for Theorem 1: $C^{k,\gamma}$-result

- The procedure is as in proof of $C^{2,\gamma}$-persistence result.
- In order to get the $L^\infty_T(\mathcal{B}^{\gamma+1,\ell}_{\infty,W})$-estimate of W, we start with estimation of $L^\infty_T(B^{\gamma-1}_{\infty,\infty})$-norm of $\partial^\ell_W \nabla^2 W$. We see

$$\partial_t(\partial^\ell_W \nabla^2 W) + u\cdot\nabla(\partial^\ell_W \nabla^2 W) = \partial^{\ell+1}_W \nabla^2 u + 2\partial^\ell_W(\nabla W\cdot\nabla^2 u) + \partial^\ell_W(\nabla^2 W\cdot\nabla u) - \partial^\ell_W(\nabla^2 u\cdot\nabla W) - \partial^\ell_W(\nabla u\cdot\nabla^2 W).$$

- Using striated estimates, we find

$$\|\partial^\ell_W \nabla^2 W(t)\|_{B^{\gamma-1}_{\infty,\infty}} \lesssim C + \int_0^t \|\partial^{\ell+1}_W \nabla^2 u\|_{B^{\gamma-1}_{\infty,\infty}}\,\mathrm{d}\tau + \int_0^t \|W\|_{\mathcal{B}^{\gamma+1,\ell}_{\infty,W}}\|\nabla u\|_{\mathcal{B}^{\gamma,\ell}_{\infty,W}}\,\mathrm{d}\tau.$$

it needs to consider $\nabla^2 u$ in $L^1_T(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})$.

Sketch of Proof for Theorem 1: $C^{k,\gamma}$-result

- In light of (9), we treat the Γ-term and θ-term separately:

$$\|\nabla^2 u\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} \le \|\nabla\nabla^\perp\Lambda^{-2}(\nabla\Gamma)\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} + \|\nabla^2\nabla^\perp\partial_1\Lambda^{-4}\theta\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})}.$$

- Estimation of θ-term. The striated estimate gives

$$\|\nabla^2\nabla^\perp\partial_1\Lambda^{-4}\theta\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} \lesssim \|\theta\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} + \int_0^t (\|W\|_{\mathcal{B}^{1,\ell}_W} + 1)\Big(\|\theta\|_{\mathcal{B}^{\gamma-1,\ell}_{\infty,W}} + \|\theta\|_{L^2}\Big)\mathrm{d}\tau.$$

Since $\partial_t(\partial^i_W\theta) + u\cdot\nabla(\partial^i_W\theta) = 0$, and using Lemma 3,

$$\|\theta\|_{L^\infty_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} \le Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\|\theta_0\|_{\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W}} \le Ce^{C(1+t)^3}.$$

Thus

$$\|\nabla^2\nabla^\perp\partial_1\Lambda^{-4}\theta\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} \le Ce^{C(1+t)^3} + Ce^{C(1+t)^3}\int_0^t \|W(\tau)\|_{\mathcal{B}^{1,\ell}_W}\mathrm{d}\tau.$$

- Estimation of Γ-term. Using striated estimate in Lemma 5,

$$\|\nabla\nabla^\perp\Lambda^{-2}(\nabla\Gamma)\|_{L^1_t(\mathcal{B}^{\gamma-1,\ell+1}_{\infty,W})} \lesssim \|\Gamma\|_{L^1_t(\mathcal{B}^{\gamma,\ell+1}_{\infty,W})} + \int_0^t \|W(\tau)\|_{\mathcal{B}^{1,\ell}_W}(\|\Gamma(\tau)\|_{\mathcal{B}^{\gamma,\ell}_{\infty,W}} + 1)\mathrm{d}\tau + 1.$$

Consider smoothing estimate of $\partial_W^{\ell+1}\Gamma$. We see

$$\partial_t(\partial_W^{\ell+1}\Gamma) + u\cdot\nabla(\partial_W^{\ell+1}\Gamma) - \Delta(\partial_W^{\ell+1}\Gamma) = [\Delta,\partial_W^{\ell+1}]\Gamma + \partial_W^{\ell+1}([\mathcal{R}_{-1}, u\cdot\nabla]\theta),$$

with

$$[\Delta,\partial_W^{\ell+1}]\Gamma = \sum_{i=0}^{\ell}\partial_W^i\Big(\Delta W\cdot\nabla\partial_W^{\ell-i}\Gamma\Big) + \sum_{i=0}^{\ell}\partial_W^i\Big(2\nabla W : \nabla^2\partial_W^{\ell-i}\Gamma\Big).$$

We finally arrive at that for $\gamma' \in (0, \min\{\gamma, 1-\frac{2}{p}\})$

$$\begin{aligned}&\|\Gamma(t)\|_{\mathcal{B}^{\gamma'-1,\ell+1}_W} + \|\Gamma\|_{L^1_t(\mathcal{B}^{\gamma'+1,\ell+1}_W)}\\ &\leq C\int_0^t\Big(\|\Gamma\|_{\mathcal{B}^{\gamma'+1,\ell}_W} + \|\nabla u\|_{L^\infty}\Big)\Big(\|W\|_{\mathcal{B}^{\gamma'+1,\ell}_W} + \|\Gamma\|_{\mathcal{B}^{\gamma'-1,\ell+1}_W}\Big)\mathrm{d}\tau + C. \qquad (15)\end{aligned}$$

- Gathering the above estimates, we finally get

$$\|W(t)\|_{\mathcal{B}^{\gamma+1,\ell}_{\infty,W}} + \|\nabla u\|_{L^1_t(\mathcal{B}^{\gamma,\ell+1}_{\infty,W})} + \|\Gamma(t)\|_{\mathcal{B}^{\gamma'-1,\ell+1}_{W}} + \|\Gamma\|_{L^1_t(\mathcal{B}^{\gamma'+1,\ell+1}_{W})}$$

$$\leq C\int_0^t \Big(\|\Gamma\|_{\mathcal{B}^{\gamma'+1,\ell}_{W}} + \|\nabla u\|_{\mathcal{B}^{\gamma,\ell}_{\infty,W}} + 1\Big)\Big(\|W\|_{\mathcal{B}^{\gamma+1,\ell}_{\infty,W}} + \|\Gamma\|_{\mathcal{B}^{\gamma'-1,\ell+1}_{W}}\Big)\mathrm{d}\tau + C.$$

- Gronwall's inequality and the induction assumption guarantee the desired result

$$\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,\ell}_{\infty,W})} + \|\nabla u\|_{L^\infty_T(\mathcal{B}^{\gamma,\ell+1}_{\infty,W})} + \|\Gamma\|_{L^\infty_T(\mathcal{B}^{\gamma'-1,\ell+1}_{W})} + \|\Gamma\|_{L^1_T(\mathcal{B}^{\gamma'+1,\ell+1}_{W})}$$

$$\leq C\exp\Big\{C\|\nabla u\|_{L^1_T(\mathcal{B}^{\gamma,\ell}_{\infty,W})} + C\|\Gamma\|_{L^1_T(\mathcal{B}^{\gamma'+1,\ell}_{W})} + CT\Big\} \lesssim_{H_\ell(T)} 1.$$

Sketch of Proof for Proposition 2

- Without loss of generality, assume $\eta_2 = 1$, $\eta_1 \approx 1$.
- Let $k \geq 3$. The proof of $B^{k+\gamma}_{\infty,1}$-persistence result for 2D INS is analogous with the $C^{k,\gamma}$-persistence result.
- The main target is to prove

$$\left(\partial_W^{k-1} W\right)(\cdot,t) \in L^\infty(0,T;B^\gamma_{\infty,1}), \quad \forall k \geq 3, \gamma \in (0,1). \tag{16}$$

We indeed will show that

$$\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,k-2}_W)} + \|u\|_{L^\infty_T(\mathcal{B}^{\gamma-1,k-1}_W)} + \|u\|_{L^1_T(\mathcal{B}^{\gamma+1,k-1}_W)} + \|(\nabla p, \partial_t u)\|_{L^1_T(\mathcal{B}^{\gamma-1,k-1}_W)} + \|\partial_W^{k-1} p\|_{L^1_T(L^\infty)} \leq H_{k-1}(T). \tag{17}$$

- We prove (17) by the induction method.
- As a starting point, (17) with $k = 2$ can be justified based on Proposition 1.
- Assume that for every $1 \leq \ell \leq k-2$,

$$\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,\ell-1}_W)} + \|u\|_{L^\infty_T(\mathcal{B}^{\gamma-1,\ell}_W)} + \|u\|_{L^1_T(\mathcal{B}^{\gamma+1,\ell}_W)} + \|(\nabla p, \partial_t u)\|_{L^1_T(\mathcal{B}^{\gamma-1,\ell}_W)} + \|\partial^\ell_W p\|_{L^1_T(L^\infty)} \leq H_\ell(T), \tag{18}$$

we intend to prove (18) with ℓ replaced by $\ell + 1$.

Sketch of Proof for Proposition 2

- In considering $\|W\|_{L^\infty_T(\mathcal{B}^{\gamma+1,\ell-1}_W)}$, we get

$$\|\partial^\ell_W W\|_{L^\infty_t(B^{1+\gamma}_{\infty,1})} \lesssim 1 + \int_0^t \|u\|_{B^{1+\gamma}_{\infty,1}} \|\partial^\ell_W W\|_{L^\infty_\tau(B^{1+\gamma}_{\infty,1})} \mathrm{d}\tau + \int_0^t \|\partial^{\ell+1}_W u\|_{B^{\gamma+1}_{\infty,1}} \mathrm{d}\tau.$$

- For $\|\partial^{\ell+1}_W u\|_{L^1_t(B^{\gamma+1}_{\infty,1})}$, consider

$$\partial_t(\partial^{\ell+1}_W u) + u\cdot\nabla(\partial^{\ell+1}_W u) - \Delta(\partial^{\ell+1}_W u) + \nabla(\partial^{\ell+1}_W p)$$
$$= (1-\rho)\Big(D_t(\partial^{\ell+1}_W u)\Big) - [\Delta, \partial^{\ell+1}_W]u + \sum_{i=0}^{\ell} \partial^i_W\Big(\nabla W\cdot\nabla\partial^{\ell-i}_W p\Big) =: F_{\ell+1} \qquad (19)$$

with $D_t = \partial_t + u\cdot\nabla$,

$$[\Delta, \partial^{\ell+1}_W]u = \sum_{i=0}^{\ell} \partial^i_W\Big(\Delta W\cdot\nabla\partial^{\ell-i}_W u\Big) + \sum_{i=0}^{\ell} \partial^i_W\Big(2\nabla W\cdot\nabla^2\partial^{\ell-i}_W u\Big).$$

- Using the striated estimates, there exists a $C_1 = C_1(D_0) > 0$ such that

$$\|\partial^{\ell+1}_W u\|_{L^\infty_t(B^{\gamma-1}_{\infty,1})} + \|\partial^{\ell+1}_W u\|_{L^1_t(B^{\gamma+1}_{\infty,1})} \leq C\Big(1 + \|\nabla\partial^{\ell+1}_W p\|_{L^1_t(B^{\gamma-1}_{\infty,1})}$$
$$+ \int_0^t \|W\|_{\mathcal{B}^{\gamma+1,\ell}_W}\Big(\|u\|_{\mathcal{B}^{\gamma+1,\ell}_W} + \|\nabla p\|_{\mathcal{B}^{\gamma-1,\ell}_W}\Big)\mathrm{d}\tau\Big) + C_1|\eta_1 - 1|\|D_t(\partial^{\ell+1}_W u)\|_{L^1_t(B^{\gamma-1}_{\infty,1})}.$$

Sketch of Proof for Proposition 2

- Concerning $\nabla\partial_W^{\ell+1}p$, letting $\mathcal{P} := \nabla\Delta^{-1}\,\mathrm{div}$, we have

$$\nabla\partial_W^{\ell+1}p = -\mathcal{P}(\partial_t\partial_W^{\ell+1}u) - \mathcal{P}\Big((\rho-1)(D_t\partial_W^{\ell+1}u)\Big) - \mathcal{P}(u\cdot\nabla\partial_W^{\ell+1}u) + \mathcal{P}(\Delta\partial_W^{\ell+1}u) + \mathcal{P}F_{\ell+1}.$$

Using the striated estimates, e.g.,

$$\|(\mathrm{Id}-\Delta_{-1})\mathcal{P}(\partial_{\mathcal{W}}^{\ell+1}\partial_t u, \partial_{\mathcal{W}}^{\ell+1}\Delta u)\|_{B^{\gamma-1}_{\infty,1}} \le C\|(\partial_t u, \Delta u)\|_{B_W^{\gamma-1,\ell}}\|W\|_{\mathcal{B}_W^{1,\ell}}.$$

we get

$$\|\nabla\partial_W^{\ell+1}p\|_{L^1_t(B^{\gamma-1}_{\infty,1})} \le C\int_0^t \Big(\|u\|_{\mathcal{B}_W^{\gamma-1,\ell+1}} + \|W\|_{\mathcal{B}_W^{\gamma+1,\ell}}\Big)\Big(\|u\|_{\mathcal{B}_W^{\gamma+1,\ell}} + \|(\nabla p, \partial_t u)\|_{\mathcal{B}_W^{\gamma-1,\ell}}\Big)\mathrm{d}\tau$$
$$+ C + C_1|\eta_0 - 1|\|D_t(\partial_W^{\ell+1}u)\|_{L^1_t(B^{\gamma-1}_{\infty,1})}. \qquad (20)$$

- Using equation (19), we infer

$$\|\partial_t(\partial_W^{\ell+1}u)\|_{L^1_t(B^{\gamma-1}_{\infty,1})} + \|D_t(\partial_W^{\ell+1}u)\|_{L^1_t(B^{\gamma-1}_{\infty,1})}$$
$$\le C\int_0^t \Big(\|u\|_{\mathcal{B}_W^{\gamma-1,\ell+1}} + \|W\|_{\mathcal{B}_W^{\gamma+1,\ell}}\Big)\Big(\|u\|_{\mathcal{B}_W^{\gamma+1,\ell}} + \|(\nabla p, \partial_t u)\|_{\mathcal{B}_W^{\gamma-1,\ell}}\Big)\mathrm{d}\tau$$
$$+ C + 6C_1|\eta_1 - 1|\|D_t(\partial_W^{\ell+1}u)\|_{L^1_t(B^{\gamma-1}_{\infty,1})}.$$

Sketch of Proof for Proposition 2

- Gathering the above estimates, and letting $20C_1|\eta_1 - 1| \leq 1$, we conclude that

$$\|W(t)\|_{\mathcal{B}_W^{1+\gamma,\ell}} + \|u(t)\|_{\mathcal{B}_W^{\gamma-1,\ell+1}} + \|u\|_{L_t^1(\mathcal{B}_W^{\gamma+1,\ell+1})} + \|(\nabla p, \partial_t u)\|_{L_t^1(\mathcal{B}_W^{\gamma-1,\ell+1})}$$
$$\leq C\int_0^t \Big(\|u(\tau)\|_{\mathcal{B}_W^{\gamma+1,\ell}} + \|(\nabla p, \partial_t u)(\tau)\|_{\mathcal{B}_W^{\gamma-1,\ell}}\Big)\Big(\|u(\tau)\|_{\mathcal{B}_W^{\gamma-1,\ell+1}} + \|W(\tau)\|_{\mathcal{B}_W^{1+\gamma,\ell}}\Big)\mathrm{d}\tau + C,$$

where $C > 0$ depends on $H_\ell(t)$.

- Gronwall's inequality and induction assumption yield that

$$\|W\|_{L_T^\infty(\mathcal{B}_W^{1+\gamma,\ell})} + \|u\|_{L_T^\infty(\mathcal{B}_W^{\gamma-1,\ell+1})} + \|u\|_{L_T^1(\mathcal{B}_W^{\gamma+1,\ell+1})} + \|(\nabla p, \partial_t u)\|_{L_t^1(\mathcal{B}_W^{\gamma-1,\ell+1})}$$
$$\leq C\exp\Big\{C\Big(\|u\|_{L_T^1(\mathcal{B}_W^{\gamma+1,\ell})} + \|(\nabla p, \partial_t u)\|_{L_T^1(\mathcal{B}_W^{\gamma-1,\ell})}\Big)\Big\} \leq H_{\ell+1}(T). \quad (21)$$

Thanks for your attention!