On the motion of interfaces of compressible and incompressible fluids with surface tension– A priori estimates

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In this talk, we consider the two phase flow. The upper flow satisfies the compressible Euler equation and the lower flow satisfies the incompressible Euler equation. This system is as following

$$\begin{cases} \partial_t \rho^+ + \mathrm{div}_{x,z}(u^+\rho^+) = 0, \quad (t,X) \in \mathbb{R}^+ \times \Omega_t^+, \\ \rho^+(\partial_t u^+ + u^+ \cdot \nabla_{x,z} u^+) + \nabla_{x,z} P^+ = 0, \quad (t,X) \in \mathbb{R}^+ \times \Omega_t^+, \\ \partial_t u^- + u^- \cdot \nabla_{x,z} u^- + \nabla_{x,z} P^- = 0, \quad (t,X) \in \mathbb{R}^+ \times \Omega_t^-, \\ \mathrm{div}_{x,z} u^- = 0, \quad (t,X) \in \mathbb{R}^+ \times \Omega_t^-, \\ (\rho^+, u^\pm)|_{t=0} = (\rho_0^+, u_0^\pm), \end{cases} \tag{1}$$

where $P^+ = (\rho^+)^2 - A$ with $A > 0$ and $X = (x,z) = (x_1, x_2, z)$. Moreover, in this paper, we assume that the initial density of fluid is away from vacuum which means that there exists a constant c_0 such that $\rho_0^+ \geq c_0 > 0$.

Boundary condition:

$$\begin{cases} \partial_t + u^{\pm} \cdot \nabla_{x,z} \quad \text{are tangent to} \quad \Gamma_t, \\ u^{+} \cdot n = u^{-} \cdot n, \\ P^{+} - P^{-} = \sigma\kappa, \end{cases} \tag{2}$$

where κ is the mean curvature of the free surface Γ_t, and n is the normal vector of Γ_t.

One Phase flow: Incompressible Euler

Results on Local well-posedness

- Without surface tension: Wu, Christodoulou-Lindblad, Zhang-Zhang, Shatah-Zeng, Coutand-Shkoller, Lannes and etc
- With surface tension: Shatah-Zeng, Ming-Zhang and etc

Results on gobal well-posedness

- Wu, Germain-Masmoudi-Shatah, Ionescu-Pausader, Alazard-Delort, Wang, Deng.......

One Phase flow: Compressible Euler

Results on Local well-posedness

- Away from vacuum: Majda, Coulombel, Secchi, Lindblad, Trakhinin, Chen, Luo.....
- Physical vacuum: Lindblad, Coutand, Shkoller, Jang, Masmoudi, Luo, Yang, Xin.....

Results on Gobal well-posedness

- Luo-Xin-Zeng, Guo-Hadzic-Jang....

Methods

- Incompressible case: div$u = 0$, all we need the estimates of the tangential derivative.
- compressible caes: use the D_t and tangential derivative to recover the normal derivative

Instability:

- Two phase flow for incompressible case: instability without surface tension.
- Two phase flow for compressible case: stability when the Mach number $M > \sqrt{2}$ for 2-D case

Main results

Now, we give the definition of energy function $E(t)$ and low-order energy function $E_l(t)$:

$$E(T) = \sup_{0\le t\le T}\Big(\sum_{i=0}^{3}\|\partial_t^{3-i}u^{\pm}\|^2_{H^i(\Omega_t^{\pm})} + \sum_{i=0}^{3}\|\partial_t^{3-i}P^{+}\|^2_{H^i(\Omega_t^{+})} + \sum_{i=1}^{3}\|\partial_t^{i}\eta\|^2_{H^{4-i}(\mathbb{R}^2)}\Big).$$

Now, we are in the position to state the main result:

Theorem

Let $u^{\pm}$ and $P^{\pm}$ be the smooth solutions to (1). For any time T, the following estimates hold

$$E(T) \le P(E(0)) + \int_0^T P(E(T)),$$

where $P(\cdot)$ is a polynomial function.

Estimates of P^-

First, we give the estimates of P^-

Lemma

Let $u^\pm$ and $P^\pm$ be the smooth solutions to (1). *Then, we have*

$$\sum_{i=0}^{2} \|\partial_t^{2-i} \nabla_{x,z} P^- \|_{H^i(\Omega_t^-)} \leq P(E(t)).$$

Proof: By the equation $(1)_3$, we have

$$\nabla_{x,z} P^- = -(\partial_t + u^- \cdot \nabla_{x,z}) u^-,$$

which implies that

$$\|\partial_t^{2-i} \nabla_{x,z} P^- \|_{H^i(\Omega^-)} \leq \|\partial_t^{2-i} \nabla_{x,z}^i (\partial_t + u^- \cdot \nabla_{x,z}) u^- \|_{L^2(\Omega^-)} \leq P(E(t)).$$

■

Improved regularity of free surface

Now, we improve the regularity of free surface because of the surface tension:

Lemma

Let $u^{\pm}$ and $P^{\pm}$ be the smooth solutions to (1)*. Then, we have*

$$\sum_{i=0}^{2} \|\partial_t^i \nabla_x \eta\|_{H^{3.5-i}(\mathbb{R}^2)} \leq P(E(t)).$$

Proof: By the boundary condition of pressure, we have that

$$\nabla_x \cdot \left(\frac{\nabla_x \eta}{\sqrt{1+|\nabla_x \eta|^2}}\right) = P^+ - P^- \in H^{2.5}(\Gamma_t) \tag{3}$$

Then, by the classical elliptic estimates, we have

$$\|\nabla_x \eta\|_{H^{3.5}(\mathbb{R}^2)} \leq P(E(t)).$$

■

Improved regularity of velocity

Next, we improve the regularity of velocity which is motivated by water wave equations. First, recalling the equation of the η, we obtain that, for $i = 1, 2$,

$$D_t^- \partial_{x_i} \eta = \widetilde{U}_i, \tag{4}$$

where $\widetilde{U}_i := \partial_{x_i} B - \nabla_x \eta \cdot \partial_{x_i} V$, $(V, B) = (u_h^-, u_3^-)|_{z=\eta} = u^-|_{z=\eta}$.

By directly calculation, we have that

$$\begin{aligned}
\widetilde{U}_i &= \Big[\partial_{x_i} v_3^- + \partial_{x_i} \eta \partial_z v_3^- - \partial_{x_j} \eta \Big(\partial_{x_i} v_j^- + \partial_{x_i} \eta \partial_z v_j^-\Big)\Big]\Big|_{z=\eta} \\
&= \Big[G(\eta) V_i + \partial_{x_i} \eta \, G(\eta) B + R^i_{\omega^-}\Big]\Big|_{z=\eta}, \\
&\triangleq \widehat{U}_i + R_{\omega^-}
\end{aligned}$$

where $G(\eta) f \triangleq \sqrt{1 + |\nabla_x \eta|^2} \partial_n f_{\mathcal{H}}$ with $f_{\mathcal{H}}$ is the harmonic extension of the function f and $R^i_{\omega^-}$ is defined by the vorticity ω^-,

Improved regularity of velocity

Introduce another new good unknown

$$U_i \triangleq V_i + T_{\partial_{x_i}\eta} B \quad (i = 1, 2), \tag{5}$$

and thus get

$$\widehat{U}_i = G(\eta)U_i + T_{G(\eta)B}\partial_{x_i}\eta \quad (i = 1, 2). \tag{6}$$

Thus, we have

$$G(\eta)U_i \quad = \quad D_t^- \partial_{x_i}\eta - R^i_{\omega^-} - T_{G(\eta)B}\partial_{x_i}\eta$$

Lemma

Let $u^\pm$ and $P^\pm$ be the smooth solutions to (1)*. Then, we have*

$$\|(D_t^-)^2 \widehat{U}_i\|_{L^2(\mathbb{R}^2)} + \|D_t^- \widehat{U}_i\|_{H^{1.5}(\mathbb{R}^2)} \leq P(E(t))$$
$$\|(D_t^-)^2 U_i\|_{H^1(\mathbb{R}^2)} + \|D_t^- U_i\|_{H^{2.5}(\mathbb{R}^2)} \leq P(E(t)).$$

Improved regularity of velocity

Lemma

Let $u^{\pm}$ and $P^{\pm}$ be the smooth solutions to (1). *Then, we have*

$$\|(D_t^-)^2(V,B)\|_{H^1(\mathbb{R}^2)} + \|D_t^-(V,B)\|_{H^{2.5}(\mathbb{R}^2)} \le P(E(t)).$$

Based on the above lemma, we can improve the regularity of P^-

Lemma

Let $u^{\pm}$ and $P^{\pm}$ be the smooth solutions to (1). *Then, we have*

$$\|D_t^-\nabla^2 P^-\|_{H^{\frac{1}{2}}(\Omega^-)} \le P(E(t)).$$

Interior estimates

- Denote $T_i = (\bar{\partial})^{3-i} D_t^i$ where $\bar{\partial} = \tau \cdot \nabla_{x,z}$ with τ is the tangential vector of the free surface.
- Acting T_i on the both sides of the $(1)_1$ to get that

$$D_t^+ T_i P^+ + 2(T_i \mathrm{div}_{x,z} u^+)(P^+ + A) = R_1^+,$$

where

$$R_1^+ := [D_t^+, T_i] P^+ + 2[P^+ + A, T_i] \mathrm{div}_{x,z} u^+.$$

- Acting T_i on the both sides of the $(1)_2$ to get that

$$\rho^+ D_t^+ (T_i u^+) + \nabla_{x,z} T_i P^+ = R_2^+,$$

where

$$R_2^+ := [\nabla_{x,z}, T_i] P^+ + [\rho^+ D_t^+, T_i] u^+.$$

Interior estimates

- By the energy method, we get

$$\frac{d}{dt}\int_{\Omega_t^+}\left(\frac{|T_iP^+|^2}{4(P^++A)}+\frac{1}{2}\rho^+|T_iu^+|^2\right)+\int_{\Gamma_t}T_iP^+(T_iu^+\cdot n)$$
$$\leq P(E(t))+\|R_1^+\|^2_{L^2(\Omega_t^+)}+\|R_2^+\|^2_{L^2(\Omega_t^+)}+\|[\mathrm{div}_{x,z},T_i]u^+\|^2_{L^2(\Omega_t^+)}.$$

- By the same argument, we give energy estimates for the u^-:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t^-}|T_iu^-|^2-\int_{\Gamma_t}T_iP^-(T_iu^-\cdot n)$$
$$\leq P(E(t))+\|R_3^-\|^2_{L^2(\Omega_t^-)}+\|[\mathrm{div}_{x,z},T_i]u^-\|^2_{L^2(\Omega_t^-)}. \tag{7}$$

where

$$R_3^-:=[D_t^-,T_i]u^-+[\nabla_{x,z},T_i]P^-.$$

Boundary estimates

- The key point of paper is to give the estimates of the following boundary term

$$\int_{\Gamma_t}\Big(T_i^+P^+\cdot(T_i^+u^+\cdot n)-T_i^-P^-\cdot(T_i^-u^-\cdot n)\Big)$$

According to the boundary condition, we have that

$$\begin{aligned}
&\int_{\Gamma_t}\Big(T_iP^+\cdot(T_iu^+\cdot n)-T_iP^-\cdot(T_iu^-\cdot n)\Big) \qquad (8)\\
=\;&\int_{\Gamma_t}(T_iP^+-T_iP^-)\cdot T_i(u^+\cdot n)+\int_{\Gamma_t}T_iP^-\cdot[T_i,n]\cdot u^- -\int_{\Gamma_t}T_iP^+\cdot[T_i,n]\cdot u^+\\
=\;&-\int_{\Gamma_t}T_i\kappa\cdot T_i\Big(\frac{\partial_t\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big)+\int_{\Gamma_t}T_iP^-\cdot[T_i,n]\cdot u^- -\int_{\Gamma_t}T_iP^+\cdot[T_i,n]\cdot u^+,
\end{aligned}$$

- According to the definition of T_i where i=0, 1, 2, 3, 4, we split the proof into the following four steps.

Boundary estimates

- $T_0 = \partial_x^3$.

$$\begin{aligned}\int_{\mathbb{R}} \partial_x^3 \kappa \cdot \partial_x^3 \Big(\frac{\partial_t \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big) &= \int_{\mathbb{R}} \partial_x^3 \Big(\partial_x \Big(\frac{\partial_x \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big)\Big) \cdot \partial_x^3 \Big(\frac{\partial_t \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big) \\ &\geq \Big\|\partial_x^3 \Big(\partial_x \Big(\frac{\partial_x \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big)\Big)\Big\|_{H^{-\frac{1}{2}}} \Big\|\partial_x^3 \Big(\frac{\partial_t \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big)\Big\|_{H^{\frac{1}{2}}} \\ &\leq P(E(t)).\end{aligned}$$

- $T_1 = \partial_x^2(\partial_t + V^+ \partial_x)$.

$$\int_{\mathbb{R}} T_1 \kappa \cdot T_1 \Big(\frac{\partial_t \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big) \leq \|(\partial_t + V^+ \partial_x)\kappa\|_{H^{\frac{3}{2}}} \Big\|T_1 \Big(\frac{\partial_t \eta}{\sqrt{1+|\partial_x \eta|^2}}\Big)\Big\|_{H^{\frac{1}{2}}} \leq P(E(t))$$

Boundary estimates

- $T_2 = \partial_x(\partial_t + V^+\partial_x)^2$.

$$
\begin{aligned}
\int_{\mathbb{R}} T_{2}\kappa \cdot T_2\Big(\frac{\partial_t\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big) &= \int_0^\infty T_{2}\kappa \cdot \partial_x\partial_t^2\Big(\frac{\partial_t\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big) \\
&+ \int_{\mathbb{R}} T_{2}\kappa \cdot \partial_x(\partial_t V^+\partial_x + 2V\partial_{tx} + (V^+\partial_x)(V^+\partial_x))\Big(\frac{\partial_t\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big) \\
&\leq \int_{\mathbb{R}} T_{2}\kappa \cdot \partial_x\partial_t^3\eta\Big(\frac{1}{\sqrt{1+|\partial_x\eta|^2}}\Big) + P(E(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{\mathbb{R}} T_{2}\kappa \cdot \partial_x\partial_t^3\eta\Big(\frac{1}{\sqrt{1+|\partial_x\eta|^2}}\Big) &\leq -\int_{\mathbb{R}} D_t^2\kappa \cdot \partial_x\Big(\partial_x\partial_t^3\eta\Big(\frac{1}{\sqrt{1+|\partial_x\eta|^2}}\Big)\Big) + P(E(t)) \\
&\leq -\int_{\mathbb{R}} D_t^2\partial_x^2\eta \cdot \partial_x^2\partial_t^3\eta\Big(\frac{1}{(1+|\partial_x\eta|^2)^2}\Big) + P(E(t)).
\end{aligned}
$$

Boundary estimates

Recalling the definition of $D_t^2 = \partial_x(\partial_t^2 + \partial_t V^+\partial_x + 2V^+\partial_{tx}^2 + (V^+\partial_x)(V^+\partial_x))$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} D_t^2\partial_x^2\eta\cdot\partial_x^2\partial_t^3\eta\left(\frac{1}{1+|\partial_x\eta|^2}\right)\\
\le\quad & -\int_{\mathbb{R}}\partial_t^2\partial_x^2\eta\cdot\partial_x^2\partial_t^3\eta\cdot\frac{1}{1+|\partial_x\eta|^2} - 2\int_{\mathbb{R}}\partial_t\partial_x^3\eta\cdot\partial_x^2\partial_t^3\eta\cdot\frac{2V^+}{1+|\partial_x\eta|^2}\\
& -\int_{\mathbb{R}}\partial_x^4\eta\cdot\partial_x^2\partial_t^3\eta\cdot\frac{(V^+)^2}{1+|\partial_x\eta|^2} + P(E(t))\\
\le\quad & -\frac{1}{2}\partial_t\int_{\mathbb{R}}|\partial_t^2\partial_x^2\eta|^2\cdot\frac{1}{1+|\partial_x\eta|^2} - 2\partial_t\int_{\mathbb{R}}\partial_t\partial_x^3\eta\cdot\partial_x^2\partial_t^2\eta\cdot\frac{2V^+}{1+|\partial_x\eta|^2}\\
& -\partial_t\int_{\mathbb{R}}\partial_x^4\eta\cdot\partial_x^2\partial_t^2\eta\cdot\frac{(V^+)^2}{1+|\partial_x\eta|^2} + P(E(t)).
\end{aligned}$$

Combining all the above estimates, we get that

$$\begin{aligned}
\int_{\mathbb{R}} T_{2\kappa}\cdot T_2\left(\frac{\partial_t\eta}{\sqrt{1+|\partial_x\eta|^2}}\right) \quad\le\quad & -\frac{1}{2}\partial_t\int_{\mathbb{R}}\frac{|\partial_t^2\partial_x^2\eta|^2}{1+|\partial_x\eta|^2} - 2\partial_t\int_{\mathbb{R}}\partial_t\partial_x^3\eta\cdot\partial_x^2\partial_t^2\eta\cdot\frac{2V^+}{1+|\partial_x\eta|^2}\\
& -\partial_t\int_{\mathbb{R}}\partial_x^4\eta\cdot\partial_x^2\partial_t^2\eta\cdot\frac{(V^+)^2}{1+|\partial_x\eta|^2} + P(E(t)).
\end{aligned}$$

Boundary estimates

- $\underline{T_3 = D_t^3}$. This part is the most difficulty part. Because there too many time derivative in the case, we can not use the Lemma 3. First, by direct calculation, we get

$$D_t \partial_x \eta = \partial_x (u^+ \cdot n \cdot \sqrt{1 + |\partial_x \eta|^2}) - V \cdot \partial_x^2 \eta$$

which implies that

$$D_t \Big(\frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}} \Big) = \frac{\partial_x (u^+ \cdot n \cdot \sqrt{1 + |\partial_x \eta|^2})}{(1 + |\partial_x \eta|^2)^{3/2}} - \frac{V \cdot \partial_x^2 \eta}{(1 + |\partial_x \eta|^2)^{3/2}}$$

Thus, we get that

$$D_t \kappa = \partial_x \frac{\partial_x (u^+ \cdot n \cdot \sqrt{1 + |\partial_x \eta|^2})}{(1 + |\partial_x \eta|^2)^{3/2}} + \partial_x \frac{V \cdot \partial_x^2 \eta}{(1 + |\partial_x \eta|^2)^{3/2}} - [\partial_x, D_t] \frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}}$$

which implies that

$$D_t^3 \kappa = D_t^2 \partial_x \frac{\partial_x (u^+ \cdot n \cdot \sqrt{1 + |\partial_x \eta|^2})}{(1 + |\partial_x \eta|^2)^{3/2}} + D_t^2 \partial_x \frac{V \cdot \partial_x^2 \eta}{(1 + |\partial_x \eta|^2)^{3/2}} - D_t^2 [\partial_x, D_t] \frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}}$$

Boundary estimates

By now, we obtain that

$$D_t^3\kappa = \frac{\partial_t^2\partial_x^2(u\cdot n)}{1+|\partial_x\eta|^2} + R_1$$

where the leading-order term in R_1 are $\partial_t^2\partial_x^3\eta$.
Next, we have

$$\begin{aligned}\int_{\mathbb{R}} D_t^3\kappa\cdot D_t^3(u^+\cdot n) &= \int_{\mathbb{R}} \frac{\partial_t^2\partial_x^2(u^+\cdot n)}{1+|\partial_x\eta|^2}\cdot D_t^3(u^+\cdot n) + \int_{\mathbb{R}} R_1\cdot D_t^3(u^+\cdot n)\\ &\geq \frac{1}{2}\partial_t\|\frac{\partial_t^2\partial_x(u^+\cdot n)}{1+|\partial_x\eta|^2}\|_{L^2}^2 - P(E(t)) + \int_{\mathbb{R}} R_1\cdot D_t^3(u^+\cdot n).\end{aligned}$$

All we left is to give the estimates of the last term of the above equality. We split the proof into the following two case.

$$\int_{\mathbb{R}} \partial_t^2\partial_x^3\eta\cdot D_t^3(u^+\cdot n)\cdot f$$

with some smooth function f.

Boundary estimates

The worst term in the first integral is that

$$\begin{aligned}\int_{\mathbb{R}} \partial_t^2\partial_x^3\eta\cdot\partial_t^3(u^+\cdot n)\cdot f &= \int_{\mathbb{R}} \partial_t^2\partial_x^3\eta\cdot\partial_t^4\eta\cdot f_1 + \int_{\mathbb{R}} \partial_t^2\partial_x^3\eta\cdot\partial_t^3\partial_x\eta\cdot f_2 \\ &= I_1 + I_2.\end{aligned}$$

where f_1 and f_2 are smooth functions. For I_1, we have

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_x\partial_t^4\eta\cdot f_1 - \int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_t^4\eta\cdot\partial_x f_1 \\ &= -\partial_t\int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_x\partial_t^3\eta\cdot f_1 + \int_{\mathbb{R}} \partial_t^3\partial_x^2\eta\cdot\partial_x\partial_t^3\eta\cdot f_1 + \int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_x\partial_t^3\eta\cdot\partial_t f_1 \\ &\quad -\partial_t\int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_t^3\eta\cdot\partial_x f_1 + \int_{\mathbb{R}} \partial_t^3\partial_x^2\eta\cdot\partial_t^3\eta\cdot\partial_x f_1 + \int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_t^3\eta\cdot\partial_x\partial_t f_1 \\ &\geq -\partial_t\int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_x\partial_t^3\eta\cdot f_1 - \partial_t\int_{\mathbb{R}} \partial_t^2\partial_x^2\eta\cdot\partial_t^3\eta\cdot\partial_x f_1 - P(E(t))\end{aligned}$$

Boundary estimates

which implies that

$$\int_0^t I_1 dt \geq -P(E(0)) - tP(E(t)) - E_l(t) - \varepsilon E(t).$$

For I_2, by the same argument as I_1, we get that

$$\int_0^t I_2 dt \geq -P(E(0)) - tP(E(t)) - E_l(t) - \varepsilon E(t).$$

Boundary estimates

Next, we give the estimates of the commutator. The main part of

$$\int_{\Gamma_t} T_i P^- \cdot [T_i, n] \cdot u^- - \int_{\Gamma_t} T_i P^+ \cdot [T_i, n] \cdot u^+$$

is that

$$\begin{aligned} \int_{\Gamma_t} D_t^3 n \cdot (D_t^3 P^- u^- - D_t^3 P^+ u^+) &= \int_{\Gamma_t} D_t^3 \kappa \cdot D_t^3 n \cdot u^- + \int_{\Gamma_t} [u] \cdot D_t^3 n \cdot D_t^3 P^- \\ &= C_1 + C_2. \end{aligned}$$

For C_2, we notice that $[u] \cdot n = 0$, we get

$$C_2 = \int_{\Gamma_t} [u \cdot \tau]\tau \cdot D_t^3 n \cdot D_t^3 P^-.$$

Boundary estimates

Here, we notice that

$$D_t\tau = (\nabla_\tau u^- \cdot n)n,$$

we get that

$$\tau \cdot D_t^3 n \sim n \cdot D_t^3 \tau = D_t^2(\nabla_\tau u^- \cdot n) \sim D_t^2(\nabla_\tau u^-) \cdot n.$$

Boundary estimates

Thus, we get that

$$
\begin{aligned}
C_2 &\leq P(E(t)) + \int_{\Omega_t^-} \operatorname{div}([u\cdot\tau] D_t^2 \nabla_\tau u^- \cdot D_t^3 P^-) \\
&\leq P(E(t)) + \int_{\Omega_t^-} [u\cdot\tau] D_t^2 \nabla_\tau u^- \cdot \nabla D_t^3 P^- \\
&\leq P(E(t)) + \int_{\Omega_t^-} [u\cdot\tau] D_t^2 \nabla_\tau u^- \cdot D_t^4 u^- \\
&\leq P(E(t)) + \partial_t \int_{\Omega_t^-} [u\cdot\tau] D_t^2 \nabla_\tau u^- \cdot D_t^3 u^- \\
&\quad + \int_{\Omega_t^-} [u\cdot\tau] D_t^3 \nabla_\tau u^- \cdot D_t^3 u^- \\
&\leq P(E(t)) + \partial_t \int_{\Omega_t^-} [u\cdot\tau] D_t^2 \nabla_\tau u^- \cdot D_t^3 u^-
\end{aligned}
$$

Thanks!