Anisotropic weighted L^q-L^r estimates of the Oseen semigroup in exterior domains, with application to the Navier-Stokes flow past a rigid body

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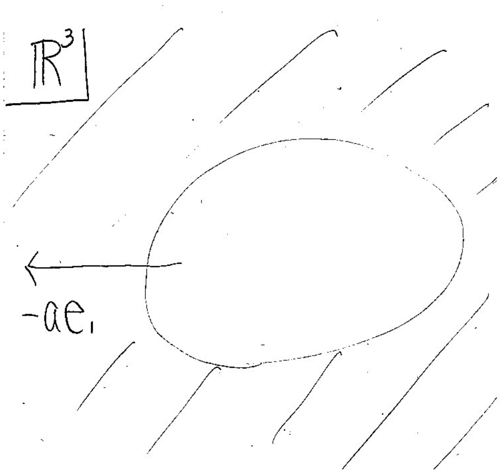
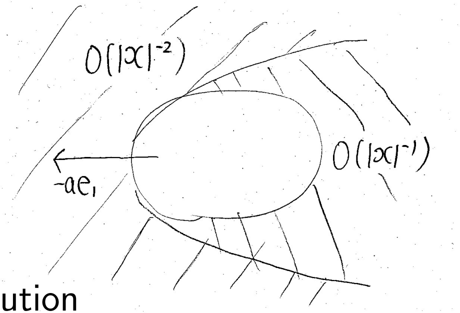


$a > 0,\ e_1 = (1, 0, 0)$

$(1 + |x|)^{\alpha}(1 + |x| - x_1)^{\beta}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$(1+|x|)^\alpha (1+|x|-x_1)^\beta$



Stationary Problem

Finn (1960's) : PR-solution

$u_s(x) = O((1+|x|)^{-1}(1+|x|-x_1)^{-1})$ as $|x| \to \infty$

Nonstationary Problem

Knightly (1979), Mizumachi (1984) :

initial perturb. has some spatial decay structure $\Rightarrow$

$$|u(x,t) - u_s| \le C(1+|x|)^{-1}(1+|x|-x_1)^{-1}$$

$$\forall t > 0, |x| >> 1$$

- Anisotropic weighted L^q-L^r estimates of the Oseen semigroup
- Stability analysis in anisotropic weighted L^q framework

Farwig-Sohr (1997) : Stokes resolvent problem in $L^q_\rho(\Omega)$ $(\Omega = \mathbb{R}^3$ or $D)$ & the Helmholtz decomposition:

$$L^q_\rho(\Omega) = L^q_{\rho,\sigma}(\Omega) \oplus \{\nabla p \in L^q_\rho(\Omega)\,;\, p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}$$

for $1 < q < \infty, \rho \in \mathscr{A}_q(\Omega)$: Muckenhoupt class

$$L^q_\rho(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega)\,;\, \int_\Omega |u(x)|^q \rho\, dx < \infty \right\},$$

$$L^q_{\rho,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^q_\rho(\Omega)}},$$

$$C^\infty_{0,\sigma}(\Omega) := \{u \in C^\infty_0(\Omega)^3\,;\, \mathrm{div}\, u = 0\}$$

Aim 1 Conditions on α, β
so that $(1+|x|)^\alpha (1+|x|-x_1)^\beta \in \mathscr{A}_q(\mathbb{R}^3)$

Theorem 1

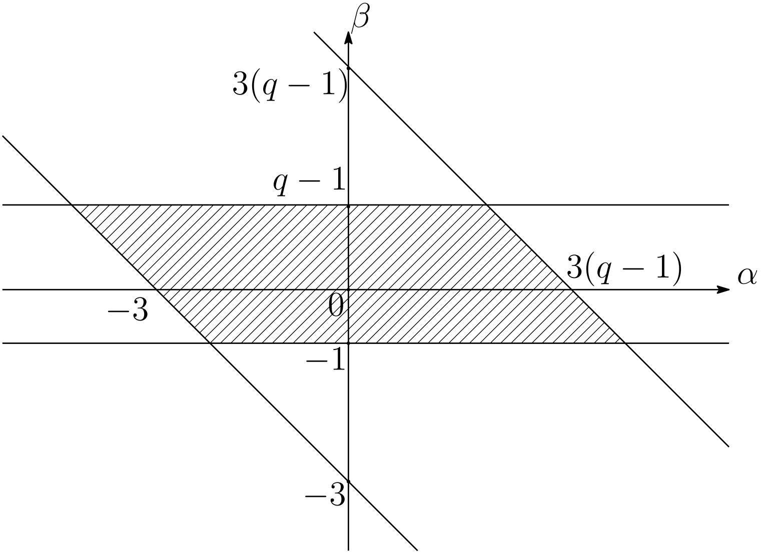
Let $1 < q < \infty$.

$\rho_{\alpha,\beta} := (1+|x|)^{\alpha}(1+|x|-x_1)^{\beta} \in \mathscr{A}_q(\mathbb{R}^3) \Leftrightarrow$

$-1 < \beta < q-1, \quad -3 < \alpha+\beta < 3(q-1).$

- Farwig (1992) : Theorem 1 with $q = 2$
- Kračmer-Novotný-Pokorný (2001) : “ $\Leftarrow$ ”

$$-1 < \beta < q-1, \quad -3 < \alpha + \beta < 3(q-1)$$



$\exists C > 0$ s.t.

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} \rho_{\alpha,\beta}\,dy\right)\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} \rho_{\alpha,\beta}^{-\frac{1}{q-1}}\,dy\right)^{q-1} \leq C$$

for all balls $B_r(x) := \{y \in \mathbb{R}^3\,;\, |y-x| < r\}$
if and only if α, β satisfy
$-1 < \beta < q-1, \;\; -3 < \alpha + \beta < 3(q-1).$

Let $\Omega = \mathbb{R}^3$ or D. Given $1 < q < \infty$ and α, β s.t.

$$-1/q < \beta < 1 - 1/q, \quad -3/q < \alpha + \beta < 3(1 - 1/q),$$

set $\rho(x) = (1 + |x|)^{\alpha q}(1 + |x| - x_1)^{\beta q}$.

- Helmholtz projection $P : L^q_\rho(\Omega) \to L^q_{\rho,\sigma}(\Omega)$
- Oseen operator $A_a : L^q_{\rho,\sigma}(\Omega) \to L^q_{\rho,\sigma}(\Omega)$ $(a \geq 0)$

$$\mathscr{D}(A_a) = W^{2,q}_\rho(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega),$$

$$A_a u = -P\left[\Delta u - a\frac{\partial u}{\partial x_1}\right].$$

$$W^{2,q}_\rho(\Omega) := \{u \in L^1_{\mathrm{loc}}(\Omega)\,;\, u, \nabla u, \nabla^2 u \in L^q_\rho(\Omega)\}$$

Aim 2 Anisotropic weighted L^q-L^r estimates of e^{-tA_a}

- Kobayashi-Shibata (1998) : If $1 < q \le r \le \infty$ $(q \neq \infty)$ (resp. $1 < q \le r \le 3$) when $k = 0$ (resp. $k = 1$). Then

$$\|\nabla^k e^{-tA_a} Pf\|_{r,D} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}} \|f\|_{q,D}$$

for $t > 0$ and $f \in L^q(D)$.

- Kobayashi-Kubo (2015) : If $1 < q \le r \le \infty$ $(q \neq \infty)$ (resp. $1 < q \le r \le 3$) when $k = 0$ (resp. $k = 1$),

$$\|(1+|x|)^\alpha \nabla^k e^{-tA_0} Pf\|_{r,D} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}} \|(1+|x|)^\alpha f\|_{q,D}$$

for $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(D)$.

Theorem 2 (Smoothing action near $t = 0$)

Given $a_0 > 0$, assume $a \in [0, a_0]$. Let $k = 0, 1$ and let $1 < q \le r \le \infty$ $(q \neq \infty)$, $\alpha, \beta \ge 0$ satisfy $\beta < 1 - 1/q$, $\alpha + \beta < 3(1 - 1/q)$. Then

$$\begin{aligned} &\|(1+|x|)^\alpha (1+|x|-x_1)^\beta \nabla^k e^{-tA_a} Pf\|_{r,D} \\ &\quad \le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}} \|(1+|x|)^\alpha (1+|x|-x_1)^\beta f\|_{q,D} \end{aligned}$$

for $t \le 1$, $f \in L^q_\rho(D)$, where C : independent of a, $\rho(x) = (1+|x|)^{\alpha q}(1+|x|-x_1)^{\beta q}$.

Theorem 3

1. Let $1 < q \le r \le \infty$ $(q \neq \infty)$ and $\alpha, \beta \ge 0$ satisfy

$\beta < \min\{1-1/q, 1/3\}$, $\alpha+\beta < \min\{3(1-1/q), 1\}$,

then

$$\begin{aligned}
&\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta} e^{-tA_a} P f\|_{r,D} \\
&\quad \le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta} f\|_{q,D} \\
&\qquad + C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})+\alpha} \|(1+|x|-x_1)^{\beta} f\|_{q,D} \\
&\qquad + C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})+\frac{\beta}{2}} \|(1+|x|)^{\alpha} f\|_{q,D} \\
&\qquad + C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})+\alpha+\frac{\beta}{2}} \|f\|_{q,D}
\end{aligned}$$

for $t \ge 1$, $f \in L^q_\rho(D)$, where C : independent of a.

2. Assume $a \in [0, a_0]$ $(a_0 > 0)$. Let $1 < q, r < \infty$ and $\alpha, \beta > 0$ satisfy

$\beta < \min\{1 - 1/q, 1/3\}, \ \alpha + \beta < \min\{3(1 - 1/q), 1\}$.

If $\alpha < 2/3$ (resp. $\alpha \geq 2/3$), we suppose

$$1 < q \leq r < \min\{3/(1 - \alpha - \beta), 3/(1 - (3\alpha)/2)\}$$

$$(\text{resp. } 1 < q \leq r < 3/(1 - \alpha - \beta)).$$

Then for $t \geq 1$, and $f \in L^q_\rho(D)$,

$$\begin{aligned} &\|(1 + |x|)^\alpha (1 + |x| - x_1)^\beta \nabla e^{-tA_a} P f\|_{r,D} \\ &\quad \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2}} \|(1 + |x|)^\alpha (1 + |x| - x_1)^\beta f\|_{q,D} \\ &\qquad + C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2} + \alpha} \|(1 + |x| - x_1)^\beta f\|_{q,D} \\ &\qquad + C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2} + \frac{\beta}{2}} \|(1 + |x|)^\alpha f\|_{q,D} \\ &\qquad + C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2} + \alpha + \frac{\beta}{2}} \|f\|_{q,D}. \end{aligned}$$

3. Assume $a \in [0, a_0]$. Let $1 < q, r < \infty$ and $\alpha > 0$ satisfy $\alpha < \min\{3(1-1/q), 1\}$ and $1 < q \le r \le 3/(1-\alpha)$. If $a > 0$, then

$$\begin{aligned}
&\|(1+|x|)^{\alpha}\nabla e^{-tA_a}Pf\|_{r,D} \\
&\quad \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|(1+|x|)^{\alpha}f\|_{q,D} \\
&\qquad + Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}+\alpha}\|f\|_{q,D}
\end{aligned}$$

for $t \ge 1$, $f \in L^q_{(1+|x|)^{\alpha q}}(D)$, and if $a = 0$, then

$$\begin{aligned}
&\|(1+|x|)^{\alpha}\nabla e^{-tA_0}Pf\|_{r,D} \\
&\quad \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|(1+|x|)^{\alpha}f\|_{q,D}
\end{aligned}$$

for $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(D)$.

Theorem 4 Assume $a \in (0, a_0]$ $(a_0 > 0)$.

Let $1 < q \le r \le \infty$ $(q \ne \infty)$ and $\alpha, \beta > 0$ satisfy $\beta < \min\{1-1/q, 1/3\}$, $\alpha+\beta < \min\{3(1-1/q), 1\}$ and $1/q - 1/r < 1/3$. Moreover, if $k = 1$ and if $\alpha < 2/3$ (resp. $\alpha \ge 2/3$), we suppose

$$1 < q \le r < \min\{3/(1-\alpha-\beta), 3/(1-(3\alpha)/2)\}$$
$$(\text{resp. } 1 < q \le r < 3/(1-\alpha-\beta)).$$

For $\varepsilon > 0$, we have

$$\begin{aligned} &\|(1+|x|)^\alpha (1+|x|-x_1)^\beta \nabla^k e^{-tA_a} Pf\|_{r,D} \\ &\le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}+\varepsilon+\frac{\alpha}{4}+\max\{\frac{\alpha}{4},\frac{\beta}{2}\}} \\ &\qquad \times \|(1+|x|)^\alpha (1+|x|-x_1)^\beta f\|_{q,D} \end{aligned}$$

for $t \ge 1$ and $f \in L^q_\rho(D)$.

Aim 3 Stability of PR-solution in anisotropic weighted Lebesgue space

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \Delta u - a\,\dfrac{\partial u}{\partial x_1} - \nabla p, & \\
\nabla \cdot u = 0, & x \in D, t > 0, \\
u|_{\partial D} = -a e_1, & t > 0, \\
u \to 0 & \text{as } |x| \to \infty, \\
u(x,0) = u_0, & x \in D.
\end{cases}
$$

• Shibata (1999), Enomoto-Shibata (2005) :
$\exists\delta, \exists\varepsilon > 0$ s.t. if $a < \delta, \|u_s - u_0\|_{3,D} < \varepsilon$
$\Rightarrow \|u(t) - u_s\|_{r,D} = O(t^{-\frac{1}{2}+\frac{3}{2r}}), \quad 3 \le r \le \infty,$
$\|\nabla u(t) - \nabla u_s\|_{3,D} = O(t^{-\frac{1}{2}})$ as $t \to \infty$

• Bae-Roh (2012) : Given some $q \le 3$ and
$0 < \alpha < 1/2$. If $a < \delta$ and
if $u_0 \in L^3_\sigma(D) \cap L^q_{(1+|x|)^{\alpha q}}(D)$ satisfies
$\|u_s - u_0\|_{3,D} < \varepsilon$

$$\Rightarrow \|(1+|x|)^\alpha (u(t) - u_s)\|_{r,D} = O(t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})+\frac{1+\alpha}{2}+\eta}),$$
$$q \le r < \infty, \forall\eta > 0.$$

$$u = v + u_s, \quad p = \phi + p_s$$

$$\rightsquigarrow v(t) = e^{-tA_a}u_0 + \int_0^t e^{-(t-\tau)A_a} P\Big[- v\cdot\nabla v - v\cdot\nabla u_s - u_s\cdot\nabla v\Big] d\tau.$$

Theorem 5

Let $\alpha, \beta \geq 0$ satisfy $\beta < 1/3 = \min\{1-1/3, 1/3\}$, $\alpha+\beta < 1 = \min\{3(1-1/3), 1\}$ and let $u_0 \in L^3_\rho(D)$, where $\rho(x) = (1+|x|)^{3\alpha}(1+|x|-x_1)^{3\beta}$. Then $\exists\delta, \exists\varepsilon > 0$ s.t. $0 < a < \delta, \|u_s - u_0\|_{3,D} < \varepsilon \Rightarrow \exists v$ enjoys

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta v(t)\|_{r,D} = O(t^{-\frac{1}{2}+\frac{3}{2r}+\alpha+\frac{\beta}{2}}) \quad 3 \leq \forall r \leq \infty,$$

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \nabla v(t)\|_{3,D} = O(t^{-\frac{1}{2}+\alpha+\frac{\beta}{2}}).$$

• Outline

1. Estimate near the boundary of D

• Local energy decay (Kobayashi-Shibata 1998) :

$$\|\partial_t e^{-tA_a} f\|_{q,D\cap B_R} + \|e^{-tA_a} f\|_{W^{2,q}(D\cap B_R)} \le Ct^{-\frac{3}{2}}\|f\|_{q,D}$$

$\forall t \ge 1,\ f \in \{f \in L^q_\sigma(D)\,;\ f(x) = 0 \text{ for } |x| \ge R\}$,
where $B_R = \{y \in \mathbb{R}^3\,;\ |y| < R\}$.

• Cut-off procedure

2. Estimate at spatial infinity

• Anisotropic weighted L^q-L^r estimates in $\mathbb{R}^3$

• Cut-off procedure

- **Step 1**

Proposition

Let $1 < q < \infty$ and $\alpha, \beta > 0$ satisfy $\alpha + \beta < 3(1 - 1/q)$. Given $s \in (\max\{(3q)/(3 + \alpha q + \beta q), (2q)/(2 + \alpha q)\}, q]$, we have

$$\|e^{-tA_a} Pf\|_{W^{2,q}(D \cap B_R)} + \|\partial_t e^{-tA_a} Pf\|_{q, D \cap B_R} \le Ct^{-\frac{3}{2s}} \|(1 + |x|)^\alpha (1 + |x| - x_1)^\beta f\|_{q,D}$$

for $t \ge 1$, $f \in L^q_\rho(D)$.