Global wellposedness for two phase problem of Navier-Stokes equations in unbounded domains

Yoshihiro Shibata

Department of Mathematics, Waseda University
Adjunct Faculty member in the Department of Mechanical Engineering and Materials Science, University of Pittsburgh

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Kato method

According to a work due to T. Kato, Math. Z. **187** (1984), 471–480, to solve the Navier-Stokes equations:

$$\partial_t \mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = 0, \quad \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega\times(0,\infty),$$
$$\mathbf{u}|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

where Ω is a domain in $\mathbb{R}^N$ $(N \geq 2)$ and Γ its boundary, if we know the existence of Stokes semigroup $\{T(t)\}_{t\geq 0}$, which is analytic, and the L_p-L_q decay estimate:

$$\|T(t)f\|_{L_q} \leq C_{p,q}t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p} \quad 1 \leq p \leq q \leq \infty;$$
$$\|\nabla T(t)f\|_{L_q} \leq C_{p,q}t^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p} \quad 1 \leq p \leq q \leq N,$$

for any $t > 1$, then, we can prove the global well-posedness for small initial data $\mathbf{u}_0 \in L_N(\Omega)$.

An idea of Kato's proof

Apply Stokes semigroup to reduce the problem to the following integral equations:

$$\mathbf{u} = \int_0^t T(t-s)P(\mathbf{u}\cdot\nabla\mathbf{u})(\cdot, s)\,ds,$$

P being the Helmholtz projection, we can show the global well-posedness by using a standard iteration scheme in the underlying space $\mathcal{I}_\epsilon$ with

$$\mathcal{I}_\epsilon = \{\mathbf{u} \in C^0((0,\infty), H^1_N(\Omega)\cap L_p(\Omega) \mid \lim_{t\to0+}\|\mathbf{u}(\cdot,t)-\mathbf{u}_0\|_{L_N(\Omega)} = 0,$$
$$\sup_{0<t<\infty}\|\mathbf{u}(\cdot,t)\|_{L_N(\Omega)} + \sup_{0<t<\infty} t^{1/2}\|\nabla\mathbf{u}(\cdot,t)\|_{L_N(\Omega)} + \sup_{0<t<\infty} t^{\frac12-\frac{N}{2p}}\|\mathbf{u}(\cdot,t)\|_{L_p(\Omega)} \leq \epsilon\}$$

with some exponent p with $N < p < \infty$ and small positive number ϵ.

Quasilinear problem in unbounded domains

We consider

$$\partial_t \mathbf{u} - A\mathbf{u} = \mathbf{F}(\mathbf{u}) \quad \text{in } \Omega \times (0, \infty), \quad B\mathbf{u}|_\Gamma = \mathbf{G}(\mathbf{u}), \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

Here, $B\mathbf{u}|_\Gamma = \mathbf{G}(\mathbf{u})$ is a boundary condition, $\mathbf{F}(\mathbf{u})$ and $\mathbf{G}(\mathbf{u})$ are nonlinear functions of $\mathbf{u}$ and its derivatives, like $\mathbf{F}(\mathbf{u}) = f(\nabla\mathbf{u})(\partial_t\mathbf{u}, \nabla^2\mathbf{u})$, $\mathbf{G}(\mathbf{u}) = g(\nabla\mathbf{u})\nabla\mathbf{u}$, quasi-linear type. I want to treat this problem in the same spirit as Kato's one.
(1) maximal L_p-L_q regularity for the time shifted linear problem:

$$\partial_t \mathbf{u} + \lambda_0\mathbf{u} - A\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times (0, \infty), \quad B\mathbf{u}|_\Gamma = \mathbf{g} \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

with some large constant $\lambda_0 > 0$. In the Stokes operator case, we have

$$\begin{aligned} &\| < t >^b \partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega))} + \| < t >^b \mathbf{u}\|_{L_p((0,\infty),H^2_q(\Omega))} \\ &\leq C\|\mathbf{u}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \| < t >^b \mathbf{f}\|_{L_p((0,\infty),L_q(\Omega))} + \| < t >^b \mathbf{g}\|_{H^{1/2}_p((0,\infty),L_q(\Omega))} \\ &\quad + \| < t >^b \mathbf{g}\|_{L_p((0,\infty),H^1_q(\Omega))}). \end{aligned}$$

Note that the Stokes operator means that an operator obtained by eliminating the pressure term.

(2) (A, B) generates an continuous analytic semigroup $\{T(t)\}_{t\geq 0}$. In the Stokes operator case, the domain is $\{\mathbf{u} \in H^2_q(\Omega) \mid \operatorname{div} \mathbf{u} = 0, \ \ B\mathbf{u}|_\Gamma = 0\}$. And, $\{T(t)\}_{t\geq 0}$ satisfies the L_p-L_q decay properties:

$$\|T(t)f\|_{L_q} \leq C_{p,q} t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p} \quad ;$$
$$\|(\nabla, \nabla^2, \partial_t)T(t)f\|_{L_q} \leq C_{p,q} t^{-\min(\frac{1}{2}+\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right), \frac{N}{2p})}\|f\|_{L_p}$$

for any $t > 1$ and $1 \leq p \leq q \leq \infty$.

In the Navier-Stoke equation case, we can solve the quasi-linear problem by using a standard iteration scheme in the underlying space $\mathcal{J}_\epsilon$ with

$$\mathcal{J}_\epsilon = \{\mathbf{u} \in H^1_p((0,\infty), L_{q_1} \cap L_{q_2}) \cap L_p((0,\infty), H^2_{q_1} \cap H^2_{q_2}) \mid \lim_{t\to 0+} \|\mathbf{u}(\cdot, t) - \mathbf{u}_0\|_{L_{q_1}\cap L_{q_2}} = 0,$$
$$\mathcal{E}(\mathbf{u}) = \| < t >^b \mathbf{u}\|_{L_\infty((0,\infty),L_{q_1}\cap L_{q_2})} + \| < t >^b \nabla\mathbf{u}\|_{L_p((0,\infty),H^1_{q_1})}$$
$$+ \| < t >^b \mathbf{u}\|_{L_p((0,\infty),H^2_{q_2})} + \| < t >^b \partial_t\mathbf{u}\|_{L_p((0,\infty),L_{q_1}\cap L_{q_2})}\}.$$

For example, we can choose p, q_1, q_2, and b as

$$\frac{1}{q_1} = \frac{1}{N} + \frac{1}{q_2}, \quad q_2 > \max(N, \frac{2N}{N-2}), \quad b > 0, \quad (1 + \frac{N}{2q_2} - b)p > 1,$$
$$bp' > 1, \quad 2/p + 1/q_2 \neq 1, \quad N \geq 3.$$

First, for $\mathbf{v} \in \mathcal{J}_\epsilon$, we consider the time shifted equation

$$\partial_t \mathbf{u}^1 + \lambda_0 \mathbf{u}^1 - A\mathbf{u}^1 = \mathbf{F}(\mathbf{v}) \quad \text{in } \Omega \times (0, \infty), \quad B\mathbf{u}|_\Gamma = \mathbf{G}(\mathbf{v}), \quad \mathbf{u}^1|_{t=0} = \mathbf{u}_0.$$

By the maximal L_p-L_q regularity,

$$\tilde{\mathcal{E}}(\mathbf{u}^1) := \sum_{q=q_1/2, q_1, q_2} (\| < t >^b \partial_t \mathbf{u}^1 \|_{L_p((0,\infty), L_q)} + \| < t >^b \mathbf{u} \|_{L_p((0,\infty), H^2_q)}) \leq C(\|\mathbf{u}_0\| + \mathcal{E}(\mathbf{v})^2).$$

Next, we consider the first compensation equations:

$$\partial_t \mathbf{u}^2 + \lambda_0 \mathbf{u}^2 - A\mathbf{u}^2 = \lambda_0 \mathbf{u}^1 \quad \text{in } \Omega \times (0, \infty), \quad B\mathbf{u}^2|_\Gamma = 0, \quad \mathbf{u}^2|_{t=0} = 0.$$

Then, by the maximal L_p-L_q regularity,

$$\tilde{\mathcal{E}}(\mathbf{u}^2) \leq C\mathcal{E}(\mathbf{u}^1) \leq C(\|\mathbf{u}_0\| + \mathcal{E}(\mathbf{v})^2).$$

Finally, we consider the second compensation equations:

$$\partial_t \mathbf{u}^3 - A\mathbf{u}^3 = \lambda_0 \mathbf{u}^2 \quad \text{in } \Omega \times (0,\infty), \quad B\mathbf{u}^3|_\Gamma = 0, \quad \mathbf{u}^3|_{t=0} = 0.$$

The role of the first compensation equations is to have $\mathbf{u}^2$ is in the domain of operators (A, B), namely in particular, $B\mathbf{u}^2|_\Gamma = 0$ for all $t > 0$.
Then, by using the L_p-L_q decay estimate for $t > 1$ and the standard analytic semigroup estimate:

$$\|\partial_t T(t)\mathbf{u}_0\|_{L_q} + \|T(t)\mathbf{u}_0\|_{H^2_q} \leq C\|\mathbf{u}_0\|_{H^2_q}$$

for $0 < t < 1$, we have

$$\mathcal{E}(\mathbf{u}^3) \leq C\tilde{\mathcal{E}}(\mathbf{u}^2) \leq C(\|\mathbf{u}_0\| + \mathcal{E}(\mathbf{v})^2).$$

Set $\mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2 + \mathbf{u}^3$, then $\mathbf{u}$ satisfies the linearized equations:

$$\partial_t \mathbf{u} - A\mathbf{u} = \mathbf{F}(\mathbf{v}) \quad \text{in } \Omega \times (0,\infty), \quad B\mathbf{u}|_\Gamma = \mathbf{G}(\mathbf{v}), \quad \mathbf{u}|_{t=0} = \mathbf{u}_0,$$

and the estimate:

$$\mathcal{E}(\mathbf{u}) \leq C(\|\mathbf{u}_0\| + \mathcal{E}(\mathbf{v})^2).$$

This implies the global wellposedness for small initial data.

Formulation of 2 phase problems

Let Ω_t^+ be a time dependent, bounded domain in $\mathbb{R}^N$ ($N \geq 2$), and $\Omega_t^- = \mathbb{R}^N \setminus \overline{\Omega_t^+}$. Let Γ_t be the boundary of Ω_t^+ and $\mathbf{n}_t$ the unit normal to Γ_t oriented from Ω_t^+ into Ω_t^-. We assume that immissible fluids $\mathcal{F}^\pm$ occupy $\Omega_t^\pm$ and the conservation of mass and the conservation of momentum are described by the Navier-Stokes equations.

Equations

$$\partial_t\rho^\pm + \operatorname{div}(\rho^\pm\mathbf{v}^\pm) = 0 \qquad \text{in } \Omega_t^\pm,$$
$$\rho^\pm(\partial_t\mathbf{v}^\pm + \mathbf{v}^\pm\cdot\nabla\mathbf{v}^\pm) - \operatorname{Div}(\mathbf{S}^\pm(\mathbf{v}^\pm) - P^\pm\mathbf{I}) = 0 \qquad \text{in } \Omega^\pm.$$

for time $t > 0$. Here, $\rho^\pm$ denotes the mass density of $\mathcal{F}^\pm$, $\mathbf{v}^\pm = (v_1^\pm, \dots, v_N^\pm)^\top$ the velocity field of $\mathcal{F}^\pm$, and $P^\pm$ the pressure field of $\mathcal{F}^\pm$. And

$$\mathbf{S}^\pm(\mathbf{v}^\pm) = \mu^\pm\mathbf{D}(\mathbf{v}^\pm) + \nu^\pm\operatorname{div}\mathbf{v}^\pm\mathbf{I}, \quad \mathbf{D}(\mathbf{v}^\pm) = \nabla\mathbf{v}^\pm + (\nabla\mathbf{v}^\pm)^\top,$$

$\mu^\pm$, $\nu^\pm$ are viscosity constants such that $\mu^\pm > 0$ and $\mu^\pm + \nu^\pm > 0$.

(1) If $\mathcal{F}^\pm$ is an incompressible, viscous fluid, we assume that $\rho^\pm = \rho_*^\pm$, which is a given, positive constant describing the mass density of the reference fluid. And so, $\operatorname{div}\mathbf{v}^\pm = 0$ and $P^\pm$ is unknown.

(2) If $\mathcal{F}^\pm$ is a comressible, viscous fluid, we assume that $\rho^\pm = \rho_*^\pm + \eta^\pm$, and $P^\pm = \mathfrak{p}^\pm(\rho^\pm)$ is a C^∞ function of $\rho^\pm$ defined on $(0,\infty)$ and assumed that $(\mathfrak{p}^\pm)'(\rho^\pm) > 0$ for any $\rho^\pm \in (0,\infty)$.

Interface conditions

$$[\mathbf{v}] = 0, \quad [(\mathbf{S}(\mathbf{v}) - P\mathbf{I})\mathbf{n}_t] = (\sigma H(\Gamma_t) - P_0)\mathbf{n}_t,$$
$$V_{\Gamma_t} = \mathbf{n}_t \cdot \mathbf{v}^+$$

on Γ_t for $t > 0$. Here,

$$[f](x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_t^+}} f^+(x) - \lim_{\substack{x \to x_0 \\ x \in \Omega_t^-}} f^-(x) \quad \text{for } x_0 \in \Gamma_t$$

which denotes the jump of functions along Γ_t.
V_{Γ_t} denotes the evolution speed of Γ_t in the $\mathbf{n}_t$ direction and $V_{\Gamma_t} = \mathbf{n}_t \cdot \mathbf{v}^+$ means the non-slip condition of Γ_t.
σ is a non-negative constant.

- When $\sigma > 0$, the σ describes the coefficient of surface tension.
- When $\sigma = 0$, we do not consider surface tension on Γ_t.

Initial conditions

$$(\rho^\pm, \mathbf{v}^\pm)|_{t=0} = (\rho_*^\pm + \theta_0^\pm, \mathbf{v}_0^\pm), \quad \Omega_t^\pm|_{t=0} = \Omega^\pm.$$

If we consider the incompressible, viscous fluid case, $\theta_0 = 0$. $\Omega^\pm$ is a reference domain, which is a uniformly smooth domain whose boundary Γ is a smooth compact hypersurface.

Local well-posedness

The local wellposedness follows from the maximal L_p-L_q regularity for the incompressible-incompressible, incompressible-compressible, and compressible-compressible case. In fact,

- Incompressible-incompressible case: Maryani and Saito Diff. Int. Eqns., **3**, 1–52, 2017 (Linear theory)
- Local well-posedness: Shibata and Saito, Chapter 3 in Fluids Under Pressure eds. T. Bodnar, G. P. Galdi, and S. Necasova, Birkhauser, 2020.
- Incompressible-Compressible: Linear theory by T. Kubo, Y. Shibata and K. Soga, Boundary Value Problem, 2014, 141,
- Local well-posedness: T. Kubo and Y. Shibata, Mathematics 2021 **9**, 621. https:// doi.org/103390/matb9060621
- Compressible-compressible, linear thoery and local well-posedness by T. Kubo, Y. Shibata and K. Soga, Discrete Contin. Dyn. Syst. **36** (2016), 3741–3774.

When two phase flows lie in a bounded container, linear thoery, local well-posedness and global well-posedness have been studied by Denisova and Solonnikov and J. Pruess and G. Simonnette.

Global well-posedness $\sigma = 0$, Incompressible-Incompressible case

In the case where $\sigma = 0$, the Lagrange coordinate is used in the standard manner. But, some difficulty to treat divergence condition, I use local Lagrange transformation. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ which equals 1 on B_R and 0 outside of B_{2R}, where R is a large number such that $\Omega^c \subset B_{R/2}$. Let $y = (y_1, \ldots, y_N)^\top \in \Omega$ be Lagrange coordinates and $\mathbf{u}^\pm(y, t)$ the velocity fields in the Lagrange coordinates. We consider Lagrange transformation:

$$x = X_{\mathbf{u}}^\pm(y, t) = y + \int_0^t \varphi(y)\mathbf{u}(y, s)\, ds. \tag{1}$$

Assume that it holds

$$\int_0^T \|\nabla(\varphi(\cdot)\mathbf{u}^\pm(\cdot, t))\|_{L_\infty(\Omega^\pm)}\, dt \leq \delta$$

with small positive constant δ as far as solutions $\mathbf{u}^\pm$ exist for $t \in (0, T)$. Under this condition and suitable regularity assumptions on $\mathbf{u}^\pm$, the map $x = X_{\mathbf{u}}^\pm(y, t)$ is diffeomorphisms from $\Omega^\pm$ onto $\Omega_t^\pm = \{x = X_{\mathbf{u}}^\pm(y, t) \mid y \in \Omega^\pm\}$.

Equations in Lagrange coordinates

Let $\mathbf{u}^\pm(y,t)$ together with $Q^\pm(y,t) = P^\pm((X^\pm_{\mathbf{u}})^{-1}(y,t),t)$ satisfy the equations:

$$\begin{aligned} \rho^\pm_* \partial_t \mathbf{u}^\pm - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}^\pm) - Q^\pm \mathbf{I}) &= \mathbf{G}^\pm(\mathbf{u}^\pm) && \text{in } \Omega \times (0,T), \\ \mathrm{div}\,\mathbf{u}^\pm = g^\pm(\mathbf{u}^\pm) &= \mathrm{div}\,\mathbf{g}(\mathbf{u}^\pm) && \text{in } \Omega \times (0,T), \\ [\mathbf{u}] = 0, \quad [(\mathbf{S}(\mathbf{u}) - Q\mathbf{I})\mathbf{n}] &= [\mathbf{H}(\mathbf{u})] && \text{on } \Gamma, \\ \mathbf{u}^\pm|_{t=0} &= \mathbf{v}^\pm_0 && \text{in } \Omega^\pm. \end{aligned} \tag{2}$$

Here, $F^\pm$, $\mathbf{G}^\pm$, H are nonlinear functions consist of products of some functions of $\mathbf{k}^\pm = \int_0^t \nabla(\varphi \mathbf{u}^\pm)\,ds$, and $\nabla^2 \mathbf{u}^\pm$, $\nabla \mathbf{k}^\pm$ and so on. Like

$$\begin{aligned} \mathbf{G}^\pm(\mathbf{u}^\pm) &= V_1(\mathbf{k}^\pm)(\partial_t \mathbf{u}^\pm, \nabla^2 \mathbf{u}^\pm) + W(\mathbf{k}^\pm) \int_0^t \nabla^2(\varphi \mathbf{u}^\pm))\,ds\,\nabla \mathbf{u}^\pm, \\ g(\mathbf{u}^\pm) &= V_2(\mathbf{k}^\pm)\nabla \mathbf{u}^\pm, \ \ \mathbf{g}(\mathbf{u}^\pm) = V_2(\mathbf{k}^\pm)\mathbf{u}^\pm, \quad \mathbf{H}(\mathbf{u}^\pm) = V_4(\mathbf{k}^\pm)\nabla \mathbf{u}^\pm. \end{aligned}$$

Here, $V_i(0) = 0$.

In fact, Jacobi matrix of the transform $x = X^{\pm}_{\mathbf{u}}(y, t)$ is $\partial x/\partial y = \mathbf{I} + \int_0^t \nabla(\varphi \mathbf{u}^{\pm})\,ds$, and so

$$\nabla_x = (\mathbf{I} + \mathbf{V}_0(\mathbf{k}^{\pm}))\nabla_y, \quad \mathbf{n}_t = \frac{(\mathbf{I} + \mathbf{V}_0(\mathbf{k}^{+})^{\top})\mathbf{n}}{|(\mathbf{I} + \mathbf{V}_0(\mathbf{k}^{+})^{\top})\mathbf{n}|}$$

with some matrices $\mathbf{V}_0$ of analytic functions of $\mathbf{k}$ with $|\mathbf{k}| < \delta$. Here, notice that $\mathbf{v}^{+} = \mathbf{v}^{-}$ on Γ_t as follows from $[\mathbf{v}] = 0$ on Γ_t.

Global well-posdedness

Assume that $N \geq 3$. Choose p, q_1, q_2, b as follows:

$$1/q_1 = 1/N + 1/q_2, \;\; q_2 > \max(N, \frac{2N}{N-2}), \quad b > 0, \quad (1 + \frac{N}{2q_2} - b)p > 1,$$
$$bp' > 1, \quad 2/p + 1/q_2 \neq 1, \;\; 0 < \sigma < 1/2.$$

Then, there exists an $\epsilon > 0$ such that for initial data $\mathbf{v}_0^\pm$ satisfying the following conditions:

$$\mathbf{v}_0^\pm \in B^{2(1-1/p)}_{q_1/2,p}(\Omega^\pm) \cap B^{2(1-1/p)}_{q_1,p}(\Omega^\pm) \cap B^{2(1-1/p)}_{q_2,p}(\Omega^\pm),$$
$$[\mathbf{v}_0] = 0, \quad [(\mathbf{S}(\mathbf{v}_0)\mathbf{n})_\tau] = 0 \quad \text{on } \Gamma, \;\; (\mathbf{d}_\tau = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n})$$

problem 3 (equations the Lagrange description) admits unique solutions $\mathbf{u}^\pm$ with

$$\mathbf{u}^\pm \in L_p((0,\infty), H^2_{q_1} \cap H^2_{q_2}(\Omega^\pm)) \cap H^1_p((0,\infty), L_{q_1} \cap L_{q_2}(\Omega^\pm))$$

with $\mathcal{E}(\mathbf{u}^\pm) \leq C\epsilon$, where C is a constant independent of $\epsilon > 0$.

$$
\begin{aligned}
\mathcal{E}(\mathbf{u}^\pm) &= \| < t >^b \ \mathbf{u}^\pm \|_{L_\infty((0,\infty), L_{q_1} \cap L_{q_2}(\Omega^\pm))} \\
&+ \| < t >^b \ (\nabla \mathbf{u}^\pm, \nabla^2 \mathbf{u}^\pm) \|_{L_p((0,\infty), L_{q_1}(\Omega^\pm))} \\
&+ \| < t >^b \ \mathbf{u}^\pm \|_{L_p((0,\infty), H^2_{q_2}(\Omega^\pm))} \\
&+ \| < t >^b \ \partial_t \mathbf{u}^\pm \|_{L_p((0,\infty), L_{q_1} \cap L_{q_2}(\Omega^\pm))}.
\end{aligned}
$$

A sketch of proof

Given $\mathbf{v}^\pm$, we consider the following linearized equations:

$$
\begin{aligned}
\rho_*^\pm \partial_t \mathbf{u}^\pm - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}^\pm) - Q^\pm \mathbf{I}) = \mathbf{G}^\pm(\mathbf{v}^\pm) &\quad \text{in } \Omega \times (0,T), \\
\mathrm{div}\,\mathbf{u}^\pm = g(\mathbf{v}^\pm) = \mathrm{div}\,\mathbf{g}(\mathbf{v}^\pm) &\quad \text{in } \Omega \times (0,T), \\
[\mathbf{u}] = 0, \quad [(\mathbf{S}(\mathbf{u}) - Q\mathbf{I})\mathbf{n}] = [\mathbf{H}(\mathbf{v})] &\quad \text{on } \Gamma, \\
\mathbf{u}^\pm|_{t=0} = \mathbf{v}_0^\pm &\quad \text{in } \Omega^\pm.
\end{aligned}
\tag{3}
$$

First, we consider time shifted equations:

$$
\begin{aligned}
\rho_*^\pm(\partial_t \mathbf{u}_1^\pm + \lambda_0 \mathbf{u}_1^\pm) - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}_1^\pm) - Q_1^\pm \mathbf{I}) = \mathbf{G}^\pm(\mathbf{v}^\pm) &\quad \text{in } \Omega \times (0,T), \\
\mathrm{div}\,\mathbf{u}_1^\pm = g(\mathbf{v}^\pm) = \mathrm{div}\,\mathbf{g}(\mathbf{v}^\pm) &\quad \text{in } \Omega \times (0,T), \\
[\mathbf{u}_1] = 0, \quad [(\mathbf{S}(\mathbf{u}_1) - Q_1\mathbf{I})\mathbf{n}] = [\mathbf{H}(\mathbf{v}^\pm)] &\quad \text{on } \Gamma, \\
(\eta_1^\pm, \mathbf{u}_1^\pm)|_{t=0} = (\theta_0^\pm, \mathbf{v}_0^\pm) &\quad \text{in } \Omega^\pm.
\end{aligned}
\tag{4}
$$

Time shifted equations

By the maximal L_p-L_q regularity results due to Mariani and Saito

$$
\begin{aligned}
&\| < t >^b \mathbf{u}_1^\pm\|_{L_p((0,\infty),H^2_{q_1/2}\cap H^2_{q_1}\cap H^2_{q_2}(\Omega^\pm))} + \| < t >^b \partial_t\mathbf{u}_1^\pm\|_{L_p((0,\infty),L_{q_1/2}\cap L_{q_1}\cap L_{q_2}(\Omega^\pm))} \\
&\leq C\{\|\mathbf{v}_0^\pm\|_{B^{2(1-1/p}_{q_1/2}\cap B^{2(1-1/p)}_{q_1,p}\cap B^{2(1-1/p)}_{q_2,p}(\Omega^\pm)} + \| < t >^b \mathbf{G}^\pm(\mathbf{v}^\pm)\|_{L_p((0,\infty),L_{q_1/2}\cap L_{q_1}\cap L_{q_2}(\Omega^\pm))} \\
&+ \| < t >^b (g(\mathbf{v}^\pm), \mathbf{H}(\mathbf{v}^\pm)\|_{L_p((0,\infty),H^1_{q_1/2}\cap H^1_{q_1}\cap H^1_{q_2}(\Omega^\pm))} \\
&+ \| < t >^b (g(\mathbf{v}^\pm), \mathbf{H}(\mathbf{v}^\pm)\|_{H^{1/2}_p((0,\infty),L_{q_1/2}\cap L_{q_1}\cap L_{q_2}(\Omega^\pm))} \\
&+ \| < t >^b \mathbf{g}(\mathbf{v}^\pm)\|_{H^1_p((0,\infty),L_{q_1/2}\cap L_{q_1}\cap L_{q_2}(\Omega^\pm))} \\
&\leq C(\|\mathbf{v}_0\| + \mathcal{E}(\theta^\pm, \mathbf{v}^\pm)^2).
\end{aligned}
$$

Here,

$$
\|\mathbf{v}_0\| = \sum_{q=q_1/2,q_1,q_2} \|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega^\pm)}.
$$

$$\| \int_0^t \nabla(\varphi \mathbf{u})\, ds\, (\mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}) \|_{L_{q_1/2}} \leq \int_0^t \|\nabla(\varphi \mathbf{u})\|_{L_{q_2}}\, ds \|\mathbf{u}\|_{H^2_{q_2}}$$
$$\leq \int_0^\infty < t >^{-bp'}\, dt \, \| < t >^b \mathbf{u}\|_{L_p((0,\infty), H^2_{q_2})} \|\mathbf{u}\|_{L_{q_2}},$$
$$\| \int_0^t \nabla(\varphi \mathbf{u})\, ds\, (\mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}) \|_{L_q} \leq \int_0^t \|\nabla(\varphi \mathbf{u})\|_{L_\infty}\, ds \|\mathbf{u}\|_{H^2_{q_2}}$$
$$\leq \int_0^\infty < t >^{-bp'}\, dt \, \| < t >^b \mathbf{u}\|_{L_p((0,\infty), H^2_{q_2})} \|\mathbf{u}\|_{H^2_{q_2}}$$

for $q = q_1, q_2$.

Secondly,

$$\begin{aligned}
\rho_*^\pm(\partial_t \mathbf{u}_2^\pm + \lambda_0 \mathbf{u}_2^\pm) - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}_2^\pm) - Q_2^\pm \mathbf{I}) &= \lambda_0 \mathbf{u}_1^\pm && \text{in } \Omega \times (0,T), \\
\mathrm{div}\, \mathbf{u}_2^\pm &= 0 && \text{in } \Omega \times (0,T), \\
[\mathbf{u}_2] = 0, \quad [(\mathbf{S}(\mathbf{u}_2) - Q_2 \mathbf{I})\mathbf{n}] &= 0 && \text{on } \Gamma, \\
\mathbf{u}_2^\pm|_{t=0} &= 0 && \text{in } \Omega^\pm.
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\| < t >^b \mathbf{u}_2^\pm\|_{L_p((0,\infty), H^2_{q_1/2} \cap H^2_{q_1} \cap H^2_{q_2}(\Omega^\pm))} + \| < t >^b \partial_t \mathbf{u}_2^\pm\|_{L_p((0,\infty), L_{q_1/2} \cap L_{q_1} \cap L_{q_2}(\Omega^\pm))} \\
&\leq C\| < t >^b \mathbf{u}_1^\pm\|_{L_p((0,\infty), L_{q_1/2} \cap L_{q_1} \cap L_{q_2}(\Omega^\pm))} \\
&\leq C(\|\mathbf{v}_0\| + \mathcal{E}(\theta^\pm, \mathbf{v}^\pm)^2).
\end{aligned}$$

Thirdly,

$$\begin{aligned}
\rho_*^\pm \partial_t \mathbf{u}_3^\pm - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}_3^\pm) - Q_3^\pm \mathbf{I}) &= \lambda_0 \mathbf{u}_2^\pm && \text{in } \Omega \times (0, T), \\
\mathrm{div}\,\mathbf{u}_3^\pm &= 0 && \text{in } \Omega \times (0, T), \\
[\mathbf{u}_3] = 0, \quad [(\mathbf{S}(\mathbf{u}_3) - Q_3 \mathbf{I})\mathbf{n}] &= 0 && \text{on } \Gamma, \\
\mathbf{u}_3^\pm|_{t=0} &= 0 && \text{in } \Omega^\pm.
\end{aligned} \tag{6}$$

By Duhamel's principle,

$$\mathbf{u}_3^\pm = \lambda_0 \int_0^t T(t-s)\mathbf{u}_2^\pm(s)\,ds.$$

Decay estimate of semigroup associated with Stokes equations,

Let $\{T(t)\}_{t\geq 0}$ be a continuouis semi-group which is analytic and associated with equations:

$$\begin{aligned} \rho_*^\pm(\partial_t \mathbf{u}^\pm - \mathrm{Div}\,(\mathbf{S}^\pm(\mathbf{u}^\pm) - Q^\pm\mathbf{I}) = 0, \quad \mathrm{div}\,\mathbf{u}^\pm &= 0 && \text{in } \Omega^\pm \times (0, T), \\ [\mathbf{u}] = 0, \quad [(\mathbf{S}(\mathbf{u}) - Q\mathbf{I})\mathbf{n}] &= 0 && \text{on } \Gamma, \\ \mathbf{u}^\pm|_{t=0} &= \mathbf{v}_0^\pm && \text{in } \Omega^\pm. \end{aligned} \tag{7}$$

We know the following L_p-L_q decay estimates:

$$\begin{aligned} \|T(t)\mathbf{u}_0\|_{L_p(\Omega^\pm)} &\leq Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\mathbf{u}_0\|_{L_q(\Omega^\pm)}, \\ \|\nabla T(t)\mathbf{u}_0\|_{L_p(\Omega^\pm)} &\leq Ct^{-\sigma_1(p,q)}\|\mathbf{u}_0\|_{L_q(\Omega^\pm)}, \\ \|(\partial_t, \nabla^2)T(t)\mathbf{u}_0\|_{L_p(\Omega^\pm)} &\leq Ct^{-\sigma_2(p,q)}\|\mathbf{u}_0\|_{L_q(\Omega^\pm)} \end{aligned}$$

for any $t > 1$ with $1 < q \leq p \leq \infty$.

$$\sigma_1(p, q) = \min(\frac{N}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{1}{2}, \frac{N}{2q}), \quad \sigma_2(p, q) = \min(\frac{N}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + 1, \frac{N}{2q}).$$

L_p-L_q decay estimate

We write

$$\mathbf{u}_3^{\pm} = \lambda_0 \int_0^t T(t-s)\mathbf{u}_2^{\pm}(s)\,ds.$$

We estimate:

$$\|\nabla^2 \mathbf{u}_3^{\pm}(\cdot, t)\|_{L_q} \leq \int_0^t \|\nabla^2 T(t-s)\mathbf{u}_2^{\pm}(s)\|_{L_q}\,ds = \Big\{\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\Big\}\|\nabla^2 T(t-s)\mathbf{u}_2^{\pm}(s)\|_{L_q}\,ds.$$

By the L_p-L_q decay estimate, we have

$$\begin{aligned}\|I(t)\|_{L_q} &\leq C\int_0^{t/2} (t-s)^{-(1+\frac{N}{2q_2})}\|\mathbf{u}_2^{\pm}(s)\|_{L_{q_1/2}}\,ds \\ &\leq C(t/2)^{-(1+\frac{N}{2q_2})}\Big(\int_0^\infty <t>^{-p'b}\ dt\Big)^{1/p'}\| <t>^b\ \mathbf{u}_2^{\pm}\|_{L_p((0,\infty),L_{q_1/2})}\end{aligned}$$

for $q = q_1$ or q_2. Here,

$$1 + \frac{N}{2q_2} = \frac{N}{2}\Big(\frac{2}{q_1} - \frac{1}{q_2}\Big) = \frac{N}{2}\Big(\frac{2}{q_1} - \frac{1}{q_1}\Big) + \frac{1}{2}.$$

Thus,

$$\| < t >^b I\|_{L_p((2,\infty),L_q)} \leq C\| < t >^b \mathbf{u}_2^\pm\|_{L_p((0,\infty),L_{q_1/2})}. \quad (\because\ (1 + \frac{N}{2q_2} - b)p > 1)$$

$$\|II(t)\|_{L_q} \leq C\int_{t/2}^{t-1} (t-s)^{-(1+\frac{N}{2q_2})}\|\mathbf{u}_2^\pm(s)\|_{L_{q_1/2}}\,ds,$$

$$< t >^b \|II(t)\|_{L_q} \leq C\int_{t/2}^{t-1} (t-s)^{-(1+\frac{N}{2q_2})} < s >^b \|\mathbf{u}_2^\pm(s)\|_{L_{q_1/2}}\,ds$$

$$\leq C\Big(\int_{t/2}^{t-1}(t-s)^{-(1+\frac{N}{2q_2})}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-(1+\frac{N}{2q_2})}(< s >^b \|\mathbf{u}_2^\pm(s)\|_{L_{q_1/2}})^p\,ds\Big)^{1/p}.$$

Thus, by Fubini's theorem

$$\Big(\int_2^\infty (< t >^b \|II(t)\|_{L_q})^p\,dt\Big)^{1/p} \leq C\Big(\int_{t/2}^\infty (t-s)^{-(1+\frac{N}{2q_2})}\,ds\Big)\| < t >^b \mathbf{u}_2^\pm\|_{L_p((0,\infty),L_{q_1/2})}.$$

Since $\mathbf{u}_2^\pm(t)$ belongs to the domain of the associated linearized equations for all $t > 0$, we have

$$\|III(t)\|_{L_q} \leq C\int_{t-1}^t \|\nabla^2 T(t-s)\mathbf{u}_2^\pm(s)\|_{L_q}\,ds \leq C\int_{t-1}^t \|\mathbf{u}_2^\pm(s)\|_{H_q^2}\,ds,$$
$$< t >^b \|III(t)\|_{L_q} \leq \Big(\int_{t-1}^t ds\Big)^{1/p'}\Big(\int_{t-1}^t (< s >^b \|\mathbf{u}_2^\pm(s)\|_{H_q^2})^p\,ds\Big)^{1/p}.$$

Thus, by Fubini's theorem

$$\Big(\int_2^\infty (< t >^b \|III(t)\|_{L_q})^p\,dt\Big)^{1/p} \leq C\Big(\int_{t-1}^t ds\Big)\| < t >^b \mathbf{u}_2^\pm\|_{L_p((0,\infty),H_q^2)}.$$

Summing up, we have obtained

$$\| < t >^b \nabla^2\mathbf{u}_3^\pm\|_{L_p((2,\infty),L_q)} \leq C(\|\mathbf{u}_2^\pm\|_{L_p((0,\infty),L_{q_1/2}\cap H_q^2)} \leq C(\|\mathbf{v}_0\| + \mathcal{E}(\mathbf{v}^\pm)^2).$$

To estimate for $t \in (0,2)$, we use the L_p-L_q maximal regularity. If we set $\mathbf{u}^\pm = \sum_{j=1}^3 \mathbf{u}_j^\pm$, we have

$$\mathcal{E}(\mathbf{u}^\pm) \leq C(\|\mathbf{v}_0\| + \mathcal{E}(\mathbf{v}^\pm)^2).$$

This implies the global well-posedness for small initial data.