Global well-posedness for a Q-tensor model of nematic liquid crystals

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Introduction

- A tensor $\mathbb{Q}$ was introduced by de Gennes (1974) in order to characterize the orientation and degree of ordering for the liquid crystal molecules.
- In the Landau-de Gennes theory, $\mathbb{Q}$ is known as a $N \times N$ traceless and symmetric matrix.
- Based on his idea, Beris and Edwards (1994) proposed the model for a viscous incompressible liquid crystal flow, which is coupled system by the Navier-Stokes equations with a parabolic-type equation describing the evolution of tensor $\mathbb{Q}$.

$\mathbb{Q}$-tensor model of nematic liquid crystals in $\mathbf{R}^N$:

$$\text{(BE)} \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla \mathfrak{p} = \Delta \mathbf{u} + \mathrm{Div}\,(\tau(\mathbb{Q}) + \sigma(\mathbb{Q})), \quad \mathrm{div}\,\mathbf{u} = 0, \\ \partial_t \mathbb{Q} + (\mathbf{u}\cdot\nabla)\mathbb{Q} - \mathbb{S}(\nabla\mathbf{u}, \mathbb{Q}) = \mathbb{H}, \\ (\mathbf{u}, \mathbb{Q})|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0). \end{cases}$$

- $\mathbf{u} = (u_1(x,t), \ldots, u_N(x,t))^T$: fluid velocity, $\mathfrak{p} = \mathfrak{p}(x,t)$: pressure, $\mathbb{Q} = \mathbb{Q}(x,t)$: order parameter of liquid crystal molecules
- $\mathrm{Div}\,\mathbb{A} = \left(\sum_{j=1}^N \partial_j A_{1j}, \ldots, \sum_{j=1}^N \partial_j A_{Nj}\right)^T$, $\partial_j = \partial/\partial x_j$
- $\tau(\mathbb{Q}) = 2\eta\mathbb{H} : \mathbb{Q}\left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right) - \eta\left[\mathbb{H}\left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right) + \left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right)\mathbb{H}\right] - \nabla\mathbb{Q}\odot\nabla\mathbb{Q}$
  $\sigma(\mathbb{Q}) = \mathbb{Q}\mathbb{H} - \mathbb{H}\mathbb{Q}$, $\eta \in \mathbf{R}$, $(\nabla\mathbb{Q}\odot\nabla\mathbb{Q})_{ij} = \sum_{\alpha,\beta=1}^N \partial_i Q_{\alpha\beta}\partial_j Q_{\alpha\beta}$
- $\mathbb{H} = L\Delta\mathbb{Q} - a\mathbb{Q} + b(\mathbb{Q}^2 - \mathrm{tr}(\mathbb{Q}^2)\mathbb{I}/N) - c\,\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}$, $L = 1$, $a, c > 0$
  $\mathbb{I}$: $N \times N$ identity matrix
- $\mathbb{S}(\nabla\mathbf{u}, \mathbb{Q}) = (\eta\mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u}))\left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right) + \left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right)(\eta\mathbf{D}(\mathbf{u}) - \mathbf{W}(\mathbf{u})) - 2\eta\left(\mathbb{Q} + \frac{1}{N}\mathbb{I}\right)\mathbb{Q} : \nabla\mathbf{u}$
- $\mathbf{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$, $\mathbf{W}(\mathbf{u}) = (\nabla\mathbf{u} - (\nabla\mathbf{u})^T)/2$,

Known results (weak solution)

- Paicu-Zarnescu (2011, 2012) : the case $\eta = 0$ or $|\eta| \ll 1$. The existence of global weak solutions in $\mathbf{R}^N$ with $N = 2, 3$. The weak-strong uniqueness in $\mathbf{R}^2$.
- Anna (2017) : the case $\eta = 0$. The existence of global weak solutions and uniqueness in $\mathbf{R}^2$ for lower regularity initial data compare with Paicu-Zarnescu.
- Huang-Ding (2015) : the case $\eta = 0$. The existence of global weak solutions with a more general energy functional in $\mathbf{R}^3$.

Remark A scalar parameter $\eta \in \mathbf{R}$ denotes the ratio between the tumbling and the aligning effects that a shear flow exert over the directors. $\eta = 0$ means that the molecules only tumble in a shear flow, but do not align.

Known results (strong solution)

- Abels-Dolzmann-Liu (2014) : For any η, they proved LWP and the existence of global weak solutions with higher regularity in time in the case of inhomogeneous mixed Dirichlet/Neumann boundary conditions in a bounded domain.
- Liu-Wang (2018) : For any η, they showed LWP to the case of anisotropic elastic energy. They improved the spatial regularity of solutions obtained in Abels-Dolzmann-Liu (2014).
- Abels-Dolzmann-Liu (2016) : the case $\eta = 0$. LWP with Dirichlet boundary condition for the classical Beris-Edwards model, which means that fluid viscosity depends on the $\mathbb{Q}$-tensor.
- Cavaterra et al. (2016) : For any η, they proved GWP in the two dimensional periodic case.
- Xiao (2017) : the case $\eta = 0$. GWP in a bounded domain. He constructed a solution in the maximal L_p-L_q regularity class.

- Schonbek-Shibata (2019) : For any η, they proved GWP in $\mathbf{R}^N$ in the maximal L_p-L_q regularity class.

(BE) $\Leftrightarrow$

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{Q} - a\mathbb{Q}) = \mathbf{f}(\mathbf{u}, \mathbb{Q}), \quad \mathrm{div}\,\mathbf{u} = 0, \\ \partial_t \mathbb{Q} - \beta \mathbf{D}(\mathbf{u}) - \Delta \mathbb{Q} + a\mathbb{Q} = \mathbb{G}(\mathbf{u}, \mathbb{Q}), \\ (\mathbf{u}, \mathbb{Q})|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0), \end{cases}$$

where $\beta = 2\eta/N$.

Their system are removed $\Delta \mathbb{Q} - a\mathbb{Q}$ from the tensor $\tau(\mathbb{Q})$. Thus, the linear part of the first equation is $\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p}$, and so linearized equations for $\mathbf{u}$ and $\mathbb{Q}$ are essentially separated.

For some η, we prove GWP for small initial data in the maximal L_p-L_q regularity class:

$$\mathbf{u} \in L_p((0,t), W_q^2(\mathbf{R}^N)) \cap W_p^1((0,t), L_q(\mathbf{R}^N)),$$
$$\mathbb{Q} \in L_p((0,t), W_q^3(\mathbf{R}^N)) \cap W_p^1((0,t), W_q^1(\mathbf{R}^N))$$

Main Theorem

Notation

- $\mathbb{S}_0 = \{\mathbb{Q} : N \times N \text{ matrix} \mid \mathrm{tr}\mathbb{Q} = 0,\ \mathbb{Q} = \mathbb{Q}^T\}$,
  $X(\mathbf{R}^N; \mathbb{S}_0) = \{\mathbb{Q} : \mathbf{R}^N \to \mathbb{S}_0 \mid \|\mathbb{Q}\|_X = \sum_{i,j}^N \|Q_{ij}\|_X < \infty\}$ $(X = W_q^\ell)$
- $J_q(\mathbf{R}^N) = \{\mathbf{u} \in L_q(\mathbf{R}^N) \mid \mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathbf{R}^N\}$
- $D_{q,p}(\mathbf{R}^N) = \{(\mathbf{u}, \mathbb{Q}) \mid \mathbf{u} \in B_{q,p}^{2(1-1/p)}(\mathbf{R}^N) \cap J_q(\mathbf{R}^N),$
  $\mathbb{Q} \in B_{q,p}^{1+2(1-1/p)}(\mathbf{R}^N; \mathbb{S}_0)\}$
- $X_{p,q,t} = \{(\mathbf{u}, \mathbb{Q}) \mid \mathbf{u} \in L_p((0,t), W_q^2(\mathbf{R}^N)) \cap W_p^1((0,t), L_q(\mathbf{R}^N)),$
  $\mathbb{Q} \in L_p((0,t), W_q^3(\mathbf{R}^N; \mathbb{S}_0)) \cap W_p^1((0,t), W_q^1(\mathbf{R}^N; \mathbb{S}_0))\}$
- $\|(\mathbf{f}, \mathbb{G})\|_{W_q^{m,\ell}(\mathbf{R}^N)} = \|\mathbf{f}\|_{W_q^m(\mathbf{R}^N)} + \|\mathbb{G}\|_{W_q^\ell(\mathbf{R}^N)}$, $\quad \|\mathbf{f}\|_{W_q^m} = \|\mathbf{f}\|_{W_q^m(\mathbf{R}^N)}$
- $\mathcal{N}(\mathbf{u}, \mathbb{Q})(T) =$
  $\sum_{q=q_1,q_2} \left(\|\langle t\rangle^b (\mathbf{u}, \mathbb{Q})\|_{L_\infty((0,T), W_q^{0,1})} + \|\langle t\rangle^b \partial_t (\mathbf{u}, \mathbb{Q})\|_{L_p((0,T), W_q^{0,1})}\right)$
  $+ \|\langle t\rangle^b \nabla(\mathbf{u}, \mathbb{Q})\|_{L_p((0,T), W_{q_1}^{1,2})} + \|\langle t\rangle^b (\mathbf{u}, \mathbb{Q})\|_{L_p((0,T), W_{q_2}^{2,3})}$,
  where $\langle t\rangle = (1 + t^2)^{1/2}$, b is given in main theorem.

Thm 1

- $N \geq 3, \ |\eta| < \dfrac{N}{2}, \quad 0 < \sigma < \dfrac{1}{2}$
- $p = 2 + \sigma, \quad q_1 = 2 + \sigma, \quad \begin{cases} q_2 \geq \dfrac{N(2+\sigma)}{N - (2+\sigma)} & \text{if } N = 3, 4, \\ q_2 > N & \text{if } N \geq 5. \end{cases}$
- $b = \dfrac{N}{2(2+\sigma)}$

Then, $\exists \epsilon > 0$ s.t. for any $(\mathbf{u}_0, \mathbb{Q}_0) \in \bigcap_{i=1}^{2} D_{q_i,p}(\mathbf{R}^N) \cap W^{0,1}_{q_1/2}(\mathbf{R}^N)$ with

$$\sum_{i=1}^{2} \|(\mathbf{u}_0, \mathbb{Q}_0)\|_{D_{q_i,p}(\mathbf{R}^N)} + \|(\mathbf{u}_0, \mathbb{Q}_0)\|_{W^{0,1}_{q_1/2}(\mathbf{R}^N)} < \epsilon^2,$$

(BE) has a unique solution $(\mathbf{u}, \mathbb{Q})$ with

$$(\mathbf{u}, \mathbb{Q}) \in X_{p,q_1,\infty} \cap X_{p,q_2,\infty}$$

satisfying the estimate

$$\mathcal{N}(\mathbf{u}, \mathbb{Q})(\infty) \leq \epsilon.$$

$\mathcal{R}$-boundedness of solution operators

Def. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$ bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for any $n \in \mathbf{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the ineq.:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)T_jf_j\|_Y^p\,du\Big\}^{\frac{1}{p}} \le C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)f_j\|_X^p\,du\Big\}^{\frac{1}{p}}.$$

The smallest such C is called $\mathcal{R}$ bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark Setting $n = 1$ in Def., we see that $\mathcal{R}$-boundedness implies uniform boundedness of the operator family $\mathcal{T}$.

(R) $$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{Q} - a\mathbb{Q}) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = 0, \\ \lambda \mathbb{Q} - \beta \mathbf{D}(\mathbf{u}) - \Delta \mathbb{Q} + a\mathbb{Q} = \mathbb{G}. \qquad (\beta = 2\eta/N) \end{cases}$$

solution formula

Taking divergence of the first equation of (R),

(1) $$\mathfrak{p} = -\beta(\mathrm{div}\,\mathrm{Div}\,\mathbb{Q} - a\Delta^{-1}\mathrm{div}\,\mathrm{Div}\,\mathbb{Q}) + \Delta^{-1}\mathrm{div}\,\mathbf{f}.$$

Inserting (1) into the first equation of (R),

(2) $$\begin{aligned} &(\lambda - \Delta)\mathbf{u} - \beta\nabla(\mathrm{div}\,\mathrm{Div}\,\mathbb{Q} - a\Delta^{-1}\mathrm{div}\,\mathrm{Div}\,\mathbb{Q}) \\ &\qquad + \beta\mathrm{Div}\,(\Delta\mathbb{Q} - a\mathbb{Q}) = \mathbf{f} - \nabla\Delta^{-1}\mathrm{div}\,\mathbf{f} =: \mathbf{g}. \end{aligned}$$

(3) the second equation of (R) $\Leftrightarrow (\lambda - (\Delta - a))\mathbb{Q} = \beta\mathbf{D}(\mathbf{u}) + \mathbb{G}.$

Applying $(\lambda - (\Delta - a))$ to (2), by (3) and $\mathrm{div}\,\mathbf{u} = 0$

$$\begin{aligned} &(\lambda - (\Delta - a))(\lambda - \Delta)\mathbf{u} + \beta^2(\Delta^2 - a\Delta)\mathbf{u} \\ &= (\lambda - (\Delta - a))\mathbf{g} + \beta\nabla(\mathrm{div}\,\mathrm{Div}\,\mathbb{G} - a\Delta^{-1}\mathrm{div}\,\mathrm{Div}\,\mathbb{G}) - \beta\mathrm{Div}\,(\Delta\mathbb{G} - a\mathbb{G}). \end{aligned}$$

Set $P(\lambda) = (\lambda - (\Delta - a))(\lambda - \Delta) + \beta^2(\Delta^2 - a\Delta)$.

$$P(\lambda)\mathbf{u} = (\lambda - (\Delta - a))(\mathbf{f} - \nabla\Delta^{-1}\mathrm{div}\,\mathbf{f}) \\ + \beta\nabla(\mathrm{div}\,\mathrm{Div}\,\mathbb{G} - a\Delta^{-1}\mathrm{div}\,\mathrm{Div}\,\mathbb{G}) - \beta\mathrm{Div}\,(\Delta\mathbb{G} - a\mathbb{G}).$$

$\therefore$ $\mathbf{u} = (u_1, \ldots, u_N)$ has form:

$$u_j = A_j(\lambda)(\mathbf{f}, \mathbb{G})$$

with

$$A_j(\lambda)(\mathbf{f}, \mathbb{G}) = \mathcal{F}^{-1}\left[\frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)}\left(\hat{f}_j - \frac{\xi_j}{|\xi|^2}\xi \cdot \hat{\mathbf{f}}\right)\right] \\ - \mathcal{F}^{-1}\left[\frac{\beta}{P(\xi, \lambda)}\left(\sum_{\ell,m=1}^{N} i\xi_j\xi_\ell\xi_m\hat{G}_{\ell m} + a\sum_{\ell,m=1}^{N}\frac{i\xi_j}{|\xi|^2}\xi_\ell\xi_m\hat{G}_{\ell m}\right)\right] \\ + \mathcal{F}^{-1}\left[\frac{\beta}{P(\xi, \lambda)}\left(\sum_{\ell=1}^{N} i\xi_\ell|\xi|^2\hat{G}_{j\ell} + a\sum_{\ell=1}^{N} i\xi_\ell\hat{G}_{j\ell}\right)\right],$$

where $P(\xi, \lambda) = (\lambda + |\xi|^2 + a)(\lambda + |\xi|^2) + \beta^2(|\xi|^4 + a|\xi|^2)$.

$\mathcal{R}$-boundedness for $\mathbf{u}$

$$
\begin{aligned}
P(\xi, \lambda) &= (\lambda + |\xi|^2 + a)(\lambda + |\xi|^2) + \beta^2(|\xi|^4 + a|\xi|^2) \\
&= (\lambda - \lambda_+)(\lambda - \lambda_-),
\end{aligned}
$$

where

$$
\begin{cases}
\lambda_+ = -(1 + \beta^2)|\xi|^2 + O(|\xi|^4), \\
\lambda_- = -(1 - \beta^2)|\xi|^2 - a + O(|\xi|^4) \ \text{ as } |\xi| \to 0,
\end{cases}
$$

$$
\lambda_\pm = -(1 \pm i|\beta|)|\xi|^2 + O(1) \ \text{ as } |\xi| \to \infty.
$$

A sector

$$
\Sigma_{\sigma,\lambda_0} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \pi - \sigma, |\lambda| \geq \lambda_0\}
$$

for $\sigma_0 < \sigma < \pi/2$ and $\lambda_0 \geq 1$, where

$$
\sigma_0 = \begin{cases}
0 & \text{if } \beta = 0, \\
\arg(1 + i|\beta|) & \text{if } \beta \neq 0.
\end{cases}
$$

Lem.(cf. Denk-Schnaubelt (2015), Enomoto-Shibata (2013), Saito (2019))
Let $1 < q < \infty$, $k(\xi, \lambda)$, $\ell(\xi, \lambda)$ be functions on $(\mathbf{R}^N \setminus \{0\}) \times \Sigma_{\sigma,0}$; $\forall\sigma \in (\sigma_0, \pi/2)$, $\forall\alpha \in \mathbf{N}_0^N$, $\exists$ positive constant $M_{\alpha,\sigma}$;

$$|\partial_\xi^\alpha k(\xi, \lambda)| \leq M_{\alpha,\sigma}|\xi|^{-|\alpha|}, \quad |\partial_\xi^\alpha \ell(\xi, \lambda)| \leq M_{\alpha,\sigma}|\xi|^{1-|\alpha|}.$$

for any $(\xi, \lambda) \in (\mathbf{R}^N \setminus \{0\}) \times \Sigma_{\sigma,0}$. Let $K(\lambda)$, $L(\lambda)$ be operators defined by

$$[K(\lambda)f](x) = \mathcal{F}^{-1}[k(\xi, \lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\sigma,0}),$$
$$[L(\lambda)f](x) = \mathcal{F}^{-1}[\ell(\xi, \lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\sigma,0}).$$

Then,

$$\exists C_{N,q}; \mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{K(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q} \max_{|\alpha| \leq N+1} M_{\alpha,\sigma}.$$

$$\exists C_{N,q}; \mathcal{R}_{\mathcal{L}(W_q^1(\mathbf{R}^N), L_q(\mathbf{R}^N))}(\{L(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q} \max_{|\alpha| \leq N+1} M_{\alpha,\sigma}.$$

$$A_j(\lambda)(\mathbf{f}, \mathbb{G}) = \mathcal{F}^{-1}\left[\frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)}\left(\hat{f}_j - \frac{\xi_j}{|\xi|^2}\xi \cdot \hat{\mathbf{f}}\right)\right] - \cdots .$$

- estimate of $P(\xi, \lambda) = (\lambda + |\xi|^2 + a)(\lambda + |\xi|^2) + \beta^2(|\xi|^4 + a|\xi|^2)$.

Lem. Let σ_0 be the same angle before. Then, $\forall \sigma \in (\sigma_0, \pi/2)$, $\forall (\xi, \lambda) \in \mathbf{R}^N \times \Sigma_{\sigma,0}$,

$$|P(\xi, \lambda)| \geq C(|\lambda|^{1/2} + |\xi|)^4$$

with some constant C independent of ξ and λ.

proof. Case: $\beta = 0$. We use

Lem. Let $0 < \epsilon < \pi/2$. Then, $\forall \lambda \in \Sigma_{\epsilon,0}$, $\forall \alpha \geq 0$,

$$|\lambda + \alpha| \geq (\sin \epsilon/2)\,(|\lambda| + \alpha).$$

Case: $\beta \neq 0$. Recall that $P(\xi, \lambda) = (\lambda - \lambda_+)(\lambda - \lambda_-)$,

$$\begin{cases} \lambda_+ = -(1+\beta^2)|\xi|^2 + O(|\xi|^4), \\ \lambda_- = -(1-\beta^2)|\xi|^2 - a + O(|\xi|^4) \ \text{ as } |\xi| \to 0, \end{cases}$$

$$\lambda_\pm = -(1 \pm i|\beta|)|\xi|^2 + O(1) \ \text{ as } |\xi| \to \infty.$$

low frequency part: we can prove by the same way as $\beta = 0$.

high frequency part: Let $\lambda = |\lambda| e^{i\theta}$ for $|\theta| \leq \pi - \sigma$, $\sigma \in (\sigma_0, \pi/2)$. Noting that $\lambda_\pm = -\sqrt{1+\beta^2}|\xi|^2 e^{\pm i\sigma_0} + O(1)$ and $\overline{\lambda_\pm} = \lambda_\mp$,

$$\begin{aligned} |\lambda - \lambda_\pm|^2 &= (\lambda - \lambda_\pm)(\overline{\lambda} - \overline{\lambda_\pm}) \\ &\geq |\lambda|^2 + (1+\beta^2)|\xi|^4 + 2\sqrt{1+\beta^2}|\lambda||\xi|^2 \cos(\theta \mp \sigma_0) - (|\lambda| + |\xi|^2 + O(1))O(1) \\ &\geq |\lambda|^2 + (1+\beta^2)|\xi|^4 - 2\sqrt{1+\beta^2}|\lambda||\xi|^2 \cos(\sigma - \sigma_0) - (|\lambda| + |\xi|^2 + O(1))O(1) \\ &\geq (1 - \cos(\sigma - \sigma_0))(|\lambda|^2 + |\xi|^4) - (|\lambda| + |\xi|^2 + O(1))O(1). \end{aligned}$$

Here, we used that $\cos(\theta \mp \sigma_0) \geq \cos\{\pi - (\sigma - \sigma_0)\} = -\cos(\sigma - \sigma_0)$. □

Let $n = 0, 1$. By $|P(\xi, \lambda)| \geq C(|\lambda|^{1/2} + |\xi|)^4$,

$$|\partial_\xi^\alpha \{(\tau\partial_\tau)^n P(\xi, \lambda)^{-1}\}| \leq C(|\lambda|^{1/2} + |\xi|)^{-4-|\alpha|}$$

for any $(\xi, \lambda) \in \mathbf{R}^N \times \Sigma_{\sigma,0}$ with $\lambda = \gamma + i\tau$.

- $\mathcal{R}$-boundedness for $\mathbf{u}$

$$u_j = A_j(\lambda)(\mathbf{f}, \mathbb{G}) = \mathcal{F}^{-1}\left[\frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)}\left(\hat{f}_j - \frac{\xi_j}{|\xi|^2}\xi \cdot \hat{\mathbf{f}}\right)\right]$$

$$- \cdots + \beta \sum_{\ell=1}^N \mathcal{F}^{-1}\left[\frac{i\xi_\ell|\xi|^2}{P(\xi, \lambda)}\hat{G}_{j\ell} + a\frac{i\xi_\ell}{P(\xi, \lambda)}\hat{G}_{j\ell}\right].$$

$|\partial_\xi^\alpha (\tau\partial_\tau)^n \{(\nabla^2, \lambda^{1/2}\nabla, \lambda)$ blue parts$\}| \leq M_{\alpha,\sigma}|\xi|^{-|\alpha|}$ $(\lambda \in \Sigma_{\sigma,\lambda_0})$.

$\therefore$ $\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^n(\nabla^2, \lambda^{1/2}\nabla, \lambda)$ blue parts $\mid \lambda \in \Sigma_{\sigma,\lambda_0}\}) \leq C_{N,q}$.

$|\partial_\xi^\alpha (\tau\partial_\tau)^n \{(\nabla^2, \lambda^{1/2}\nabla, \lambda)$ red parts$\}| \leq M_{\alpha,\sigma}|\xi|^{1-|\alpha|}$ $(\lambda \in \Sigma_{\sigma,0})$.

$\therefore$ $\mathcal{R}_{\mathcal{L}(W_q^1(\mathbf{R}^N), L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^n(\nabla^2, \lambda^{1/2}\nabla, \lambda)$ red parts $\mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q}$.

Thm 2 Let $1 < q < \infty$. Then, $\forall\sigma \in (\sigma_0, \pi/2)$, $\exists\lambda_0 = \lambda_0(\sigma) \geq 1$ and operators

$$A(\lambda) \in \mathrm{Hol}(\Sigma_{\sigma,\lambda_0}, \mathcal{L}(W_q^{0,1}(\mathbf{R}^N), W_q^2(\mathbf{R}^N)))$$
$$B(\lambda) \in \mathrm{Hol}(\Sigma_{\sigma,\lambda_0}, \mathcal{L}(W_q^{0,1}(\mathbf{R}^N), W_q^3(\mathbf{R}^N)))$$

s.t. $\forall\lambda = \gamma + i\tau \in \Sigma_{\sigma,\lambda_0}$, $\forall\mathbf{f} \in L_q(\mathbf{R}^N)$, $\forall\mathbb{G} \in W_q^1(\mathbf{R}^N)$,

$$\mathbf{u} = A(\lambda)(\mathbf{f}, \mathbb{G}), \quad \mathbb{Q} = B(\lambda)(\mathbf{f}, \mathbb{G})$$

are unique sol. of problem (R) and

$$\mathcal{R}_{\mathcal{L}(W_q^{0,1}(\mathbf{R}^N), L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^n \mathcal{S}_\lambda A(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q},$$
$$\mathcal{R}_{\mathcal{L}(W_q^{0,1}(\mathbf{R}^N), L_q(\mathbf{R}^N)\times W_q^1(\mathbf{R}^N))}(\{(\tau\partial_\tau)^n \mathcal{T}_\lambda B(\lambda) \mid \lambda \in \Sigma_{\sigma,\lambda_0}\}) \leq C_{N,q}$$

for $n = 0, 1$, where $\mathcal{S}_\lambda \mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$,
$\mathcal{T}_\lambda\mathbb{Q} = (\nabla^3\mathbb{Q}, \lambda^{1/2}\nabla^2\mathbb{Q}, \lambda\mathbb{Q})$, and $C_{N,q}$ is a constant independent of λ.

generation of semigroup

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{Q} - a\mathbb{Q}) = 0, \quad \mathrm{div}\, \mathbf{v} = 0, \\ \partial_t \mathbb{Q} - \beta \mathbf{D}(\mathbf{u}) - \Delta \mathbb{Q} + a\mathbb{Q} = O, \\ (\mathbf{u}, \mathbb{Q})|_{t=0} = (\mathbf{f}, \mathbb{G}). \end{cases}$$

Set

$$X_q(\mathbf{R}^N) = J_q(\mathbf{R}^N) \times W_q^1(\mathbf{R}^N).$$

Let $\mathcal{A}$ be a linear operator defined by

$$\mathcal{A}(\mathbf{u}, \mathbb{Q}) = (P\Delta \mathbf{u} - \beta P \mathrm{Div}\,(\Delta \mathbb{Q} - a\mathbb{Q}), \beta \mathbf{D}(\mathbf{u}) + \Delta \mathbb{Q} - a\mathbb{Q})$$

for $(\mathbf{u}, \mathbb{Q}) \in D(\mathcal{A})$, where P denotes solenoidal projection and

$$D(\mathcal{A}) = (W_q^2(\mathbf{R}^N) \cap J_q(\mathbf{R}^N)) \times W_q^3(\mathbf{R}^N).$$

By $\mathcal{R}$-boundedness, $\forall \lambda \in \Sigma_{\sigma,\lambda_0}$ and $\forall (\mathbf{f}, \mathbb{G}) \in X_q(\mathbf{R}^N)$,

$$|\lambda| \|(\mathbf{u}, \mathbb{Q})\|_{W_q^{0,1}} + |\lambda|^{1/2} \|(\nabla \mathbf{u}, \nabla^2 \mathbb{Q})\|_{L_q} + \|(\mathbf{u}, \mathbb{Q})\|_{W_q^{2,3}} \leq C \|(\mathbf{f}, \mathbb{G})\|_{W_q^{0,1}}.$$

$\Rightarrow$ $\mathcal{A}$ generates continuous analytic semigroup $\{e^{\mathcal{A}t}\}_{t \geq 0}$ on $X_q(\mathbf{R}^N)$.

$\Rightarrow$ $\|e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W_p^{2,3}} + \|\partial_t e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W_p^{0,1}} \leq C \|(\mathbf{f}, \mathbb{G})\|_{W_p^{2,3}} \quad (0 < t < 2).$

maximal L_p-L_q regularity

Let X and Y be Banach spaces.

$\mathcal{D}(\mathbf{R}, X)$: the space of X valued C^∞ functions with compact support.

$\mathcal{S}(\mathbf{R}, X)$: the space of X valued rapidly decreasing functions.

$\mathcal{S}'(\mathbf{R}, X) = \mathcal{L}(\mathcal{S}(\mathbf{R}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbf{R}, \mathcal{L}(X, Y))$, we define an operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbf{R}, X) \to \mathcal{S}'(\mathbf{R}, Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]] \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbf{R}, X)).$$

The operator-valued Fourier multiplier theorem (Weis, 2001)

Let X and Y be UMD Banach spaces and $1 < p < \infty$.
Let $M \in C^1(\mathbf{R}\backslash\{0\}, \mathcal{L}(X, Y))$ s.t.

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{M(\tau) \mid \tau \in \mathbf{R}\backslash\{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau M'(\tau) \mid \tau \in \mathbf{R}\backslash\{0\}\}) = \kappa_1 < \infty.$$

Then, the operator $T_M\phi$ is extended to a bounded linear operator from $L_p(\mathbf{R}, X)$ into $L_p(\mathbf{R}, Y)$. Moreover,

$$\|T_M f\|_{L_p(\mathbf{R},Y)} \leq C(\kappa_0 + \kappa_1)\|f\|_{L_p(\mathbf{R},X)} \quad (f \in L_p(\mathbf{R}, X)).$$

Let $U = (\mathbf{u}, \mathbb{Q})$, $\mathbf{F} = (P\mathbf{f}, \mathbb{G})$.

$$\partial_t \mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F} \text{ in } \mathbf{R}^N \text{ for } t > 0, \quad \mathbf{U}|_{t=0} = 0.$$

Let $e^{-\gamma_1 t}\mathbf{F} \in L_p(\mathbf{R}_+, W_q^{0,1}(\mathbf{R}^N))$ and let $\mathbf{F}_0$ be the zero extension of $\mathbf{F}$ to $t < 0$. We consider

$$\partial_t \mathbf{U}_1 - \mathcal{A}\mathbf{U}_1 = \mathbf{F}_0 \quad \text{ in } \mathbf{R}^N \text{ for } t \in \mathbf{R}.$$

Applying Laplace transform yields $\lambda\mathcal{L}[\mathbf{U}_1] - \mathcal{A}\mathcal{L}[\mathbf{U}_1] = \mathcal{L}[\mathbf{F}_0]$.

$$\therefore \mathcal{L}[\mathbf{U}_1](\lambda) = (A(\lambda)\mathcal{L}[\mathbf{F}_0](\lambda), B(\lambda)\mathcal{L}[\mathbf{F}_0](\lambda)),$$

where $A(\lambda)$ and $B(\lambda)$ given in Thm 2. Let $\lambda = \gamma + i\tau$. By Laplace inverse transform, we define $\mathbf{U}_1$ by

$$\mathbf{U}_1(\cdot, t) = \mathcal{L}^{-1}[(A(\lambda)\mathcal{L}[\mathbf{F}_0](\lambda), B(\lambda)\mathcal{L}[\mathbf{F}_0](\lambda))](t) \;\; (\gamma \geq \gamma_1).$$

$$\Leftrightarrow e^{-\gamma t}\mathbf{U}_1 = \mathcal{F}^{-1}[(A(\lambda)\mathcal{F}[e^{-\gamma t}\mathbf{F}_0](\tau), B(\lambda)\mathcal{F}[e^{-\gamma t}\mathbf{F}_0](\tau))].$$

By Thm 2 and the operator-valued Fourier multiplier theorem,

$$\|e^{-\gamma t}\mathbf{U}_1\|_{L_p(\mathbf{R}, W_q^{2,3}(\mathbf{R}^N))} + \|e^{-\gamma t}\partial_t \mathbf{U}_1\|_{L_p(\mathbf{R}, W_q^{0,1}(\mathbf{R}^N))} \leq C\|e^{-\gamma t}\mathbf{F}\|_{L_p(\mathbf{R}_+, W_q^{0,1}(\mathbf{R}^N))}.$$

L_p-L_q decay estimates of $\{e^{\mathcal{A}t}\}_{t\geq 0}$

$$\partial_t \mathbf{U} - \mathcal{A}\mathbf{U} = 0 \text{ in } \mathbf{R}^N \text{ for } t > 0, \quad \mathbf{U}|_{t=0} = (\mathbf{f}, \mathbb{G}).$$

$$\mathbf{u} = \frac{1}{2\pi i}\mathcal{F}^{-1}\left[\int_\Gamma e^{\lambda t}\frac{\lambda + |\xi|^2 + a}{P(\xi,\lambda)}\hat{\mathbf{f}}\, d\lambda\right] + \cdots .$$

$$\|\nabla^j e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}(\|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_q(\mathbf{R}^N)} + \|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)})$$

$$\|\partial_t e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}(\|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_q(\mathbf{R}^N)} + \|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)})$$

for $t \geq 1$, $1 < q < 2 \leq p < \infty$, $j = 0, 1, 2$.

Outline of proof

underlying space:

$$\mathcal{I}_{T,\epsilon} = \{(\mathbf{u}, \mathbb{Q}) \in X_{p,q_1,T} \cap X_{p,q_2,T} \mid (\mathbf{u}, \mathbb{Q})|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0), \quad \mathcal{N}(\mathbf{u}, \mathbb{Q})(T) \leq \epsilon, \\ \sup_{0<t<T} \|\mathbb{Q}(\cdot, t)\|_{L_\infty(\mathbf{R}^N)} \leq 1\}.$$

Given $(\mathbf{u}, \mathbb{Q}) \in \mathcal{I}_{T,\epsilon}$, let $(\mathbf{v}, \mathbb{P})$ be a solution to the equation:

$$\begin{cases} \partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{P} - a\mathbb{P}) = \mathbf{f}(\mathbf{u}, \mathbb{Q}), \quad \mathrm{div}\, \mathbf{v} = 0, \\ \partial_t \mathbb{P} - \beta \mathbf{D}(\mathbf{v}) - \Delta \mathbb{P} + a\mathbb{P} = \mathbb{G}(\mathbf{u}, \mathbb{Q}), \\ (\mathbf{v}, \mathbb{P})|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0). \end{cases}$$

In order to prove $(\mathbf{v}, \mathbb{P}) \in \mathcal{I}_{T,\epsilon}$, we check

$$\mathcal{N}(\mathbf{v}, \mathbb{P})(T) \leq C\epsilon^2.$$

Set $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$.
$(\mathbf{v}_1, \mathbb{P}_1)$ satisfies time shifted equations:

$$\begin{cases} \partial_t \mathbf{v}_1 + \lambda_1 \mathbf{v}_1 - \Delta \mathbf{v}_1 + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{P}_1 - a\mathbb{P}_1) = \mathbf{f}(\mathbf{u}, \mathbb{Q}), \quad \mathrm{div}\,\mathbf{v}_1 = 0, \\ \partial_t \mathbb{P}_1 + \lambda_1 \mathbb{P}_1 - \beta \mathbf{D}(\mathbf{v}_1) - \Delta \mathbb{P}_1 + a\mathbb{P}_1 = \mathbb{G}(\mathbf{u}, \mathbb{Q}), \\ (\mathbf{v}_1, \mathbb{P}_1)|_{t=0} = (0, O). \end{cases}$$

$(\mathbf{v}_2, \mathbb{P}_2)$ satisfies compensation equations:

$$\begin{cases} \partial_t \mathbf{v}_2 - P\Delta \mathbf{v}_2 + \beta P \mathrm{Div}\,(\Delta \mathbb{P}_2 - a\mathbb{P}_2) = \lambda_1 \mathbf{v}_1, \\ \partial_t \mathbb{P}_2 - \beta \mathbf{D}(\mathbf{v}_2) - \Delta \mathbb{P}_2 + a\mathbb{P}_2 = \lambda_1 \mathbb{P}_1, \\ (\mathbf{v}_2, \mathbb{P}_2)|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0), \end{cases}$$

where P is solenoidal projection.

Analysis of time shifted equations

- maximal L_p-L_q regularity

(L)
$$\begin{cases} \partial_t \mathbf{v}_1 + \lambda_1 \mathbf{v}_1 - \Delta \mathbf{v}_1 + \nabla \mathfrak{p} + \beta \mathrm{Div}\,(\Delta \mathbb{P}_1 - a\mathbb{P}_1) = \mathbf{f}, \quad \mathrm{div}\, \mathbf{v}_1 = 0, \\ \partial_t \mathbb{P}_1 + \lambda_1 \mathbb{P}_1 - \beta \mathbf{D}(\mathbf{v}_1) - \Delta \mathbb{P}_1 + a\mathbb{P}_1 = \mathbb{G}, \\ (\mathbf{v}_1, \mathbb{P}_1)|_{t=0} = (0, O). \end{cases}$$

Thm. Let $1 < p, q < \infty$, $b \geq 0$. Then, $\exists \lambda_1 \geq 1$; $\forall (\mathbf{f}, \mathbb{G}) \in L_p((0,T), W_q^{0,1}(\mathbf{R}^N))$, (L) has a unique sol. $(\mathbf{v}_1, \mathbb{P}_1) \in X_{p,q,T}$;

$$\|\langle t\rangle^b \partial_t (\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T), W_q^{0,1}(\mathbf{R}^N))} + \|\langle t\rangle^b (\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T), W_q^{2,3}(\mathbf{R}^N))}$$
$$\leq C \|\langle t\rangle^b (\mathbf{f}, \mathbb{G})\|_{L_p((0,T), W_q^{0,1}(\mathbf{R}^N))}.$$

- estimates for nonlinear terms in L_{q_1}, L_{q_2}, $L_{q_1/2}$
- Summing up, $q_2 > N$, $1 - bp < 0 \Rightarrow$

$$\|\langle t\rangle^b \partial_t (\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T), W_q^{0,1}(\mathbf{R}^N))} + \|\langle t\rangle^b (\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T), W_q^{2,3}(\mathbf{R}^N))}$$
$$\leq C \mathcal{N}(\mathbf{u}, \mathbb{Q})(T)^2 \ \ (q = q_1/2, q_1, q_2).$$

Analysis of compensation equations

- Recall that
  $\mathcal{A}(\mathbf{u},\mathbb{Q}) = (P\Delta\mathbf{u} - \beta P\mathrm{Div}\,(\Delta\mathbb{Q} - a\mathbb{Q}),\ \beta\mathbf{D}(\mathbf{u}) + \Delta\mathbb{Q} - a\mathbb{Q})$.
- compensation equations $\Leftrightarrow$

$$\partial_t(\mathbf{v}_2,\mathbb{P}_2) - \mathcal{A}(\mathbf{v}_2,\mathbb{P}_2) = (\lambda_1\mathbf{v}_1,\lambda_1\mathbb{P}_1),\quad (\mathbf{v}_2,\mathbb{P}_2)|_{t=0} = (\mathbf{u}_0,\mathbb{Q}_0),$$

  By Duhamel's principle

$$(\mathbf{v}_2,\mathbb{P}_2) = e^{\mathcal{A}t}(\mathbf{u}_0,\mathbb{Q}_0) + \lambda_1 \underline{\int_0^t e^{\mathcal{A}(t-s)}(\mathbf{v}_1,\mathbb{P}_1)(\cdot,s)\,ds}.$$

- Case: $t > 2$. estimates of spatial derivatives:

$$\|\nabla^j \underline{\qquad}\|_{W_q^{0,1}} \le \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^{t}\right)\|\nabla^j e^{\mathcal{A}(t-s)}(\mathbf{v}_1,\mathbb{P}_1)\|_{W_q^{0,1}}\,ds.$$

$$\|\nabla^j e^{\mathcal{A}t}(\mathbf{f},\mathbb{G})\|_{W_p^{0,1}(\mathbf{R}^N)} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}(\|(\mathbf{f},\mathbb{G})\|_{W_q^{0,1}(\mathbf{R}^N)} + \|(\mathbf{f},\mathbb{G})\|_{W_p^{0,1}(\mathbf{R}^N)})$$

for $t \ge 1$, $1 < q < 2 \le p < \infty$, $j = 0, 1, 2$.

$$\|e^{\mathcal{A}t}(\mathbf{f},\mathbb{G})\|_{W_p^{2,3}(\mathbf{R}^N)} \le C\|(\mathbf{f},\mathbb{G})\|_{W_p^{2,3}(\mathbf{R}^N)} \quad \text{for } 0 < t < 2.$$

$$\|\nabla^j \underline{\quad\quad}\|_{W^{0,1}_q} \le C\|\langle t\rangle^b(\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T),W^{0,1}_{q_1/2})} + \|\langle t\rangle^b(\mathbf{v}_1, \mathbb{P}_1)\|_{L_p((0,T),W^{2,3}_q)}$$
$$\le C\mathcal{N}(\mathbf{u}, \mathbb{Q})(T)^2 \quad \text{for } \begin{cases} j = 1, 2 & \text{if } q = q_1, \\ j = 0, 1, 2 & \text{if } q = q_2. \end{cases}$$

Remark

$$\text{decay rate of } \{e^{\mathcal{A}t}\}_{t\ge 0} = \frac{N}{2}\left(\frac{2}{q_1} - \frac{1}{q_1}\right) = \frac{N}{2(2+\sigma)}$$
$$= b,$$

so that we can get decay estimates for $\nabla(\mathbf{v}_2, \mathbb{P}_2)$ and $\nabla^2(\mathbf{v}_2, \mathbb{P}_2)$ in L_{q_1}.

Combining estimates for time shifted eq. and estimates for compensation eq., we have

$$\mathcal{N}(\mathbf{v}, \mathbb{P})(T) \le C\epsilon^2.$$

Estimates for nonlinear terms

- estimates for nonlinear terms in L_{q_1}, L_{q_2},
  $\mathbb{G}(\mathbf{u}, \mathbb{Q}) = -(\mathbf{u}\cdot\nabla)\mathbb{Q} - 2\xi(\mathbb{Q} + \mathbb{I}/N)\mathbb{Q} : \nabla\mathbf{u} + \cdots .$
  $$q_2 > N \Rightarrow \|\langle t\rangle^b \mathbb{Q}(\mathbb{Q} : \nabla\mathbf{u})\|_{L_p((0,T),L_q)} \le C\left(\int_0^T \langle t\rangle^{bp}\|\mathbb{Q}\|_{W^1_{q_2}}^p \|\nabla\mathbf{u}\|_q^p\,dt\right)^{1/p}$$
  $$\le C\|\mathbb{Q}\|_{L_\infty(W^1_{q_2})}\|\langle t\rangle^b\nabla\mathbf{u}\|_{L_p(L_q)} \le C\mathcal{N}(\mathbf{u}, \mathbb{Q})(T)^2.$$
- estimates for nonlinear terms in $L_{q_1/2}$ (for preparation)
  $\mathbb{G}(\mathbf{u}, \mathbb{Q}) = \cdots + b(\mathbb{Q} - \mathrm{tr}(\mathbb{Q}^2)\mathbb{I}/N).$

$$\|\langle t\rangle^b\mathbb{Q}^2\|_{L_p((0,T),L_{q_1/2})} = \left(\int_0^T \langle t\rangle^{bp}\|\mathbb{Q}^2\|_{L_{q_1/2}}\,dt\right)^{1/p}$$
$$= \left(\int_0^T \langle t\rangle^{-bp}\langle t\rangle^{2bp}\|\mathbb{Q}\|_{L_{q_1}}^{2p}\,dt\right)^{1/p}$$
$$\le \left(\sup_{0<t<T}\|\mathbb{Q}\|_{L_{q_1}}\right)^2\left(\int_0^T \langle t\rangle^{-bp}\,dt\right)^{1/p}$$
$$\le C\mathcal{N}(\mathbf{u}, \mathbb{Q})(T)^2 \quad \text{if } 1 - bp < 0.$$

Analysis of compensation equations

- Recall that

$$\mathcal{A}(\mathbf{u}, \mathbb{Q}) = (P\Delta\mathbf{u} - \beta P\mathrm{Div}\,(\Delta\mathbb{Q} - a\mathbb{Q}),\ \beta\mathbf{D}(\mathbf{u}) + \Delta\mathbb{Q} - a\mathbb{Q})$$

  generates a semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ satisfying L_p-L_q estimates.
- compensation equations $\Leftrightarrow$

$$\partial_t(\mathbf{v}_2, \mathbb{P}_2) - \mathcal{A}(\mathbf{v}_2, \mathbb{P}_2) = (\lambda_1\mathbf{v}_1, \lambda_1\mathbb{P}_1),\ \ (\mathbf{v}_2, \mathbb{P}_2)|_{t=0} = (\mathbf{u}_0, \mathbb{Q}_0),$$

  By Duhamel's principle

$$(\mathbf{v}_2, \mathbb{P}_2) = e^{\mathcal{A}t}(\mathbf{u}_0, \mathbb{Q}_0) + \lambda_1 \int_0^t e^{\mathcal{A}(t-s)}(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)\,ds.$$

- Let $(\tilde{\mathbf{v}}_2, \tilde{\mathbb{P}}_2) = \int_0^t e^{\mathcal{A}(t-s)}(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)\,ds$. We consider estimates of spatial derivatives for $(\tilde{\mathbf{v}}_2, \tilde{\mathbb{P}}_2)$ in L_p-L_q, where $q = q_1, q_2$, $p = q_1 = 2 + \sigma$, $q_2 > N$.
- What is a suitable assumption for b (decay rate)?

- Case: $t > 2$.

$$\begin{aligned}&\|\nabla^j(\tilde{\mathbf{v}}_2, \tilde{\mathbb{P}}_2)\|_{W^{0,1}_q(\mathbf{R}^N)} \\ &\le \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^{t}\right) \|\nabla^j e^{\mathcal{A}(t-s)}(\mathbf{v}_1, \mathbb{P}_1)\|_{W^{0,1}_q}\, ds \\ &=: I_q(t) + II_q(t) + III_q(t).\end{aligned}$$

$$\|\nabla^j e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p} \le C t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}(\|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_q(\mathbf{R}^N)} + \|(\mathbf{f}, \mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)})$$

for $t \ge 1$, $1 < q < 2 \le p < \infty$, $j = 0, 1, 2$.

$$\|e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W^{2,3}_p(\mathbf{R}^N)} \le C\|(\mathbf{f}, \mathbb{G})\|_{W^{2,3}_p(\mathbf{R}^N)} \quad \text{for } 0 < t < 2.$$

- We use decay estimates with $(p, q) = (q_1, q_1/2), (q_2, q_1/2)$.
- $[[(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)]] := \|(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)\|_{W^{0,1}_{q_1/2}} + \sum_{q=q_1,q_2} \|(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)\|_{W^{2,3}_q}$,

$$\tilde{\mathcal{N}}(\mathbf{v}_1, \mathbb{P}_1)(T) = \left(\int_0^T (\langle t\rangle^b [[(\mathbf{v}_1, \mathbb{P}_1)(\cdot, t)]])^p\, dt\right)^{1/p} \le C\epsilon^2.$$

- $I_q(t)$

$$\|\nabla^j e^{\mathcal{A}t}(\mathbf{f},\mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}(\|(\mathbf{f},\mathbb{G})\|_{W^{0,1}_q(\mathbf{R}^N)} + \|(\mathbf{f},\mathbb{G})\|_{W^{0,1}_p(\mathbf{R}^N)})$$

for $t \ge 1,\ 1 < q < 2 \le p < \infty,\ j = 0,1,2.$

Let $\ell = \dfrac{N}{2(2+\sigma)} + \dfrac{1}{2}$. Decay rates of $\{e^{\mathcal{A}t}\}_{t\ge 0}$:

$$(p,q) = (q_1, q_1/2) \Rightarrow \frac{N}{2}\left(\frac{2}{q_1} - \frac{1}{q_1}\right) + \frac{j}{2} = \frac{N}{2(2+\sigma)} + \frac{j}{2} \ge \ell \ \ (j = 1,2),$$

$$(p,q) = (q_2, q_1/2) \Rightarrow \frac{N}{2}\left(\frac{2}{q_1} - \frac{1}{q_2}\right) + \frac{j}{2} > \frac{2N}{2(2+\sigma)} - \frac{N-(2+\sigma)}{2(2+\sigma)} + \frac{j}{2}$$

$$= \frac{N}{2(2+\sigma)} + \frac{1}{2} + \frac{j}{2} \ge \ell \ \ (j = 0,1,2)$$

if $q_1 = 2+\sigma$, $\begin{cases} q_2 \ge \dfrac{N(2+\sigma)}{N-(2+\sigma)} & (N = 3,4), \\ q_2 > N\left(> \dfrac{N(2+\sigma)}{N-(2+\sigma)}\right) & (N \ge 5). \end{cases}$

$$
\begin{aligned}
I_q(t) &\le C\int_0^{t/2}(t-s)^{-\ell}[[(\mathbf{v}_1,\mathbb{P}_1)(\cdot,s)]]\,ds \\
&\le C(t/2)^{-\ell}\left(\int_0^\infty \langle s\rangle^{-p'b}\,ds\right)^{1/p'}\left(\int_0^T\left(\langle s\rangle^{b}[[(\mathbf{v}_1,\mathbb{P}_1)(\cdot,s)]]\right)^p\,ds\right)^{1/p} \\
&\le Ct^{-\ell}\tilde{\mathcal{N}}(\mathbf{v}_1,\mathbb{P}_1)(T) \quad \text{if } 1-p'b<0.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_2^T\left(\langle t\rangle^b I_q(t)\right)^p\,dt &\le C\int_2^T\langle t\rangle^{-(\ell-b)p}\,dt\;\tilde{\mathcal{N}}(\mathbf{v}_1,\mathbb{P}_1)(T)^p \\
&\le C\tilde{\mathcal{N}}(\mathbf{v}_1,\mathbb{P}_1)(T)^p \quad \text{if } 1-(\ell-b)p<0.
\end{aligned}
$$

Remark $1-p'b<0,\ 1-(\ell-b)p<0 \Rightarrow 1=\dfrac{1}{p}+\dfrac{1}{p'}<(\ell-b)+b=\ell.$

Here,

$$\ell=\frac{N}{2(2+\sigma)}+\frac{1}{2}>1, \text{ but } \frac{N}{2(2+\sigma)}<1 \text{ if } N=3,4,$$

so that we can get decay estimates for $\nabla(\mathbf{u},\mathbb{Q})$ and $\nabla^2(\mathbf{u},\mathbb{Q})$ in L_{q_1}.

- decay rate b

  By $\ell = \dfrac{N}{2(2+\sigma)} + \dfrac{1}{2}$,

$$1 - (\ell - b)p < 0 \Rightarrow b < \ell - \frac{1}{p} = \frac{N}{2(2+\sigma)} + \frac{1}{2} - \frac{1}{2+\sigma}.$$

$$\therefore \; b := \frac{N}{2(2+\sigma)}.$$

We check $1 - p'b < 0$. By $p = 2 + \sigma$, $p' = \dfrac{2+\sigma}{1+\sigma}$.

$$\therefore 1 - p'b = 1 - \frac{2+\sigma}{1+\sigma}\frac{N}{2(2+\sigma)} = 1 - \frac{N}{2(1+\sigma)} < 1 - \frac{3}{2(1+\sigma)} < 0$$

if $0 < \sigma < 1/2$.

- $II_q(t)$

Using $\langle t\rangle^b \le C\langle s\rangle^b$ for $t/2 < s < t-1$ and Hölder's inequality,

$$
\begin{aligned}
&\langle t\rangle^b II_q(t) \\
&\quad \le C\left(\int_{t/2}^{t-1} (t-s)^{-\ell}\,ds\right)^{1/p'} \left(\int_{t/2}^{t-1} (t-s)^{-\ell}\left(\langle s\rangle^b[[(\mathbf{v}_1,\mathbb{P}_1)(\cdot,s)]]\right)^p\,ds\right)^{1/p}.
\end{aligned}
$$

By Fubini's theorem,

$$
\begin{aligned}
\int_2^T \left(\langle t\rangle^b II_q(t)\right)^p\,dt &\le C\int_1^T \int_{s+1}^{2s} (t-s)^{-\ell}\,dt\left(\langle s\rangle^b[[(\mathbf{v}_1,\mathbb{P}_1)(\cdot,s)]]\right)^p\,ds \\
&\le C\tilde{\mathcal{N}}(\mathbf{v}_1,\mathbb{P}_1)(T)^p.
\end{aligned}
$$

- $III_q(t)$

(4) $\|e^{\mathcal{A}t}(\mathbf{f}, \mathbb{G})\|_{W_p^{2,3}(\mathbf{R}^N)} \leq C\|(\mathbf{f}, \mathbb{G})\|_{W_p^{2,3}(\mathbf{R}^N)}$ for $0 < t < 2$.

$$III_q(t) \leq C \int_{t-1}^{t} \|(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)\|_{W_q^{2,3}(\mathbf{R}^N)}\, ds \leq C \int_{t-1}^{t} [[(\mathbf{v}_1, \mathbb{P}_1)(\cdot, s)]]\, ds.$$

Employing the same method as in the estimate of $II_q(t)$,

$$\int_2^T \left(< t >^b III_q(t)\right)^p dt \leq C\tilde{\mathcal{N}}(\mathbf{v}_1, \mathbb{P}_1)(T)^p.$$

- Case: $0 < t < \min(2, T)$. We use (4).
- Summing up,

$$\|\langle t\rangle^b \nabla(\tilde{\mathbf{v}}_2, \tilde{\mathbb{P}}_2)\|_{L_p((0,T),W_{q_1}^{1,2}(\mathbf{R}^N))} + \|\langle t\rangle^b (\tilde{\mathbf{v}}_2, \tilde{\mathbb{P}}_2)\|_{L_p((0,T),W_{q_2}^{2,3}(\mathbf{R}^N))} \leq C\tilde{\mathcal{N}}(\mathbf{v}_1, \mathbb{P}_1)(T).$$