The two-phase Stokes flow by capillarity in the plane

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Based on joint works with Georg Prokert (TU Eindhoven)

Outline

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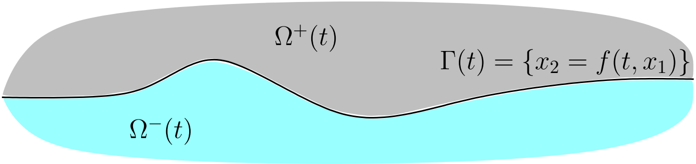
1. The mathematical model

2. The main result

The quasistationary Stokes flow



- 2D geometry, unbounded fluid domains

$$\Omega^{\pm}(t) := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \gtrless f(t, x_1)\}$$

- Far away from the origin the flow is almost stationary (and the interface almost flat)
- $\mu^{\pm} > 0$ is the viscosity of the fluid $\pm$
- $\sigma > 0$ is surface tension coefficient at the moving boundary $\Gamma(t)$
- Gravity effects are neglected

The equations of motion

Incompressible Stokes equations

$$\left.\begin{aligned} \mu^{\pm}\Delta v^{\pm} - \nabla p^{\pm} &= 0 \\ \mathrm{div}\, v^{\pm} &= 0 \end{aligned}\right\} \qquad \text{in } \Omega^{\pm}(t)$$

Notation:

- $v^{\pm}$ is velocity of the fluid $\pm$
- $p^{\pm}$ is the pressure of the fluid $\pm$

Boundary conditions

Boundary condition on $\Gamma(t)$:

$$\left.\begin{array}{rcl} [v] & = & 0 \\ [T_\mu(v,p)]\nu & = & -\sigma\kappa\nu \end{array}\right\} \quad \text{in } \Gamma(t)$$

where the stress tensor $T_\mu(v,p)$ is given by

$$T_\mu(v,p) := -pI_2 + \mu(\nabla v + (\nabla v)^\top)$$

Given any function $z : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ we set $z^\pm := z|_{\Omega^\pm}$ and

$$[z](x) := \lim_{\Omega^+ \ni y \to x} z^+ - \lim_{\Omega^- \ni y \to x} z^-, \qquad x \in \Gamma$$

Notation

- $\kappa = \kappa(t)$ is the curvature of $\Gamma(t)$
- $\nu = \nu(t)$ outer unit normal at $\partial\Omega^-(t)$

Boundary conditions

Far-field boundary conditions:

$$\begin{cases} f(t,\xi) & \to & 0 \quad \text{for } |\xi| \to \infty \\ (v^{\pm}, p^{\pm})(x) & \to & 0 \quad \text{for } |x| \to \infty \end{cases}$$

The normal velocity V_n of $\Gamma(t)$ is given by

$$V_n \;=\; v^{\pm} \cdot \nu \quad \text{on } \Gamma(t)$$

We arrive at the following system

$$\left.\begin{array}{rcll} \mu^\pm \Delta v^\pm - \nabla p^\pm & = & 0 & \text{in } \Omega^\pm(t) \\ \operatorname{div} v^\pm & = & 0 & \text{in } \Omega^\pm(t) \\ [v] & = & 0 & \text{on } \Gamma(t) \\ [T_\mu(v,p)]\nu & = & -\sigma\kappa\nu & \text{on } \Gamma(t) \\ f(\xi) & \to & 0 & \text{for } |\xi| \to \infty \\ (v^\pm, p^\pm)(x) & \to & 0 & \text{for } |x| \to \infty \\ V_n & = & v^\pm \cdot \nu & \text{on } \Gamma(t) \end{array}\right\} \quad \text{for } t > 0$$

with initial condition $f(0) = f_0$

Properties:

- Moving boundary problem (with unknowns: f, $v^\pm$, $p^\pm$)
- Nonlinear (quasilinear) evolution problem
- The first 4 equations are linear PDEs for v and/or p with constant coefficients

Literature

- Günther & Prokert '97 (arbitrary space dimension): H^k-initial data with $k \geq 3 + \frac{n+1}{2}$
- Solonnikov '99: $C^{3+\alpha}$-initial data (in 3D)
- Friedman & Reitich '02: Initial data in H^5 (in 2D), resp. H^6 (in 3D)
- Prüss & Simonett '16 (arbitrary space dimension): Initial data in $W_p^{2+\mu-2/p}$, $p \in (1, \infty)$, and $1 \geq \mu > \frac{n+2}{p}$

Remark:

- In these references the phase space is always embedded in C^2
- If $f = f(t, \xi)$ is a solution to the Stokes flow, then also

$$f_\lambda(t, \xi) = \lambda^{-1} f(\lambda t, \lambda \xi), \qquad \lambda > 0$$

In 2D this identifies $H^{3/2}(\mathbb{R})$ as a critical space

Remarks

- Goal: Establish well-posedness in all subcritical spaces $H^s(\mathbb{R})$ with $s \in (3/2, 2)$
- Given $f \in H^s(\mathbb{R})$ with $s \in (3/2, 2)$, the curvature operator

$$\kappa = \frac{f''}{(1 + f'^2)^{3/2}}$$

is not a function

- With respect to the term $\kappa\nu$, an important observation in our analysis is the following relation

$$\omega\kappa\nu = g', \qquad g := (g_1, g_2) := \Big(-\frac{f'^2}{\omega + \omega^2}, \frac{f'}{\omega}\Big),$$

where

$$\omega = \sqrt{1 + f'^2}$$

The term g is fully nonlinear!

Outline

The main result

Theorem (M & Prokert '20, '21)

Let $s \in (3/2, 2)$ *be given. Then:*

(i) (Well-posedness) *Given* $f_0 \in H^s(\mathbb{R})$*, there exists a unique maximal solution* $(f, v^\pm, p^\pm)$ *such that*

- $f = f(\cdot; f_0) \in \mathrm{C}([0, T_+), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_+), H^{s-1}(\mathbb{R}))$,
- $v^\pm(t) \in \mathrm{C}^2(\Omega^\pm(t)) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)})$, $p^\pm(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)})$ *for all* $t \in (0, T_+)$,
- $v(t)^\pm|_{\Gamma(t)} \in H^2(\mathbb{R})^2$ *for all* $t \in (0, T_+)$,

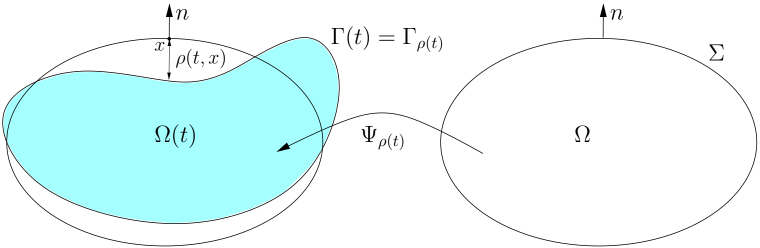
where $T_+ = T_+(f_0) \in (0, \infty]$ *(+ continuous dependence on data).*

(ii) (Parabolic smoothing)

(iia) *The map* $[(t, \xi) \mapsto f(t, \xi)] : (0, T_+) \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a* C^∞*-function.*

(iib) *For any* $k \in \mathbb{N}$*, we have* $f \in \mathrm{C}^\infty((0, T_+), H^k(\mathbb{R}))$.

(iii) $T_+(f_0) = \infty$ *if for each* $T > 0$ *we have* $\sup\limits_{[0,T]\cap[0,T_+(f_0))} \|f(t)\|_{H^s} < \infty$.



- Use Hanzawa transform (or Lagrangian coordinates) to transform the problem on a fixed (smooth) domain Ω with boundary Σ
- New unknowns: ρ and pulled back variables $u := \Psi_\rho^* v$ and $q := \Psi_\rho^* p$

Drawbacks:

- The differential operators have coefficients depending on ρ: $\Delta \leftrightarrow \mathcal{A}(\rho)$
- The solutions u and q of the transformed Stokes equations depend in an intricate way on ρ

Drawbacks:

- At the end the problem is formulated as a nonlinear and nonlocal evolution equation
$$\frac{d\rho}{dt} = \mathcal{F}(\rho)$$
and $\mathcal{F}$ involves solution operators to ρ-dependent equations
- The solution operators are difficult to handle in unbounded geometries (with asymptotic boundary conditions)
- In order to define the solution operators rather restrictive regularity and smallness assumptions on ρ are needed

An alternative approach when considering the full-space problem:

- Use potential theory to determine v and p explicitly in terms of f via an integral representation (Badea & Duchon '98)
- At the end the problem is formulated as a nonlinear and nonlocal evolution equation (see also Muskat problem)
$$\frac{df}{dt} = \mathcal{F}(f)$$
and $\mathcal{F}(f)$ is defined explicitly in terms of singular integral operators

Solving the fixed time problem

Theorem (M & Prokert '20, '21)

Given $f \in H^3(\mathbb{R})$, there exists a unique solution (v, p) to

$$\left.\begin{array}{rcll} \mu^\pm \Delta v^\pm - \nabla p^\pm & = & 0 & \text{in } \Omega^\pm \\ \mathrm{div}\, v^\pm & = & 0 & \text{in } \Omega^\pm \\ [v] & = & 0 & \text{on } \Gamma \\ [T_\mu(v,p)]\nu & = & -\sigma\kappa\nu & \text{on } \Gamma \\ f(\xi) & \to & 0 & \text{for } |\xi| \to \infty \\ (v^\pm, p^\pm)(x) & \to & 0 & \text{for } |x| \to \infty \end{array}\right\}$$

such that $(\mu^+ - \mu^-) v^\pm|\Gamma \in H^2(\mathbb{R})^2$.

Remarks:

- The proof is technical (especially when $\mu^+ \neq \mu^-$)
- v is determined in terms of an explicit contour integral

Formal computations leading to v in the case $\mu^+ = \mu^- =: \mu$

Using Stokes' formula it holds in $\mathcal{D}'(\mathbb{R}^2)$ that

$$\mu\Delta v - \nabla p = -\sigma\kappa\nu\delta_\Gamma =: (F_1, F_2)$$

Hence

$$v = \mathcal{U}^k * F_k, \qquad p = \mathcal{P}^k * F_k$$

where

$$(\mathcal{U}^k, \mathcal{P}^k) : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \times \mathbb{R}, \qquad k = 1, 2,$$

are the fundamental solutions to the Stokes equations in $\mathbb{R}^2$ [Ladyzhenskaya, '63]

$$\mathcal{U}^k = (\mathcal{U}_1^k, \mathcal{U}_2^k)^\top,$$

$$\mathcal{U}_j^k(y) = -\frac{1}{4\pi\mu}\left(\delta_{jk}\ln\frac{1}{|y|} + \frac{y_j y_k}{|y|^2}\right), \quad j = 1,\ 2,$$

$$\mathcal{P}^k(y) = -\frac{1}{2\pi}\frac{y_k}{|y|^2}, \quad y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$$

Formal computations leading to v in the case $\mu^+ = \mu^- =: \mu$

Hence, we have for $x \in \mathbb{R}^2 \setminus \Gamma$

$$v(x) = \int_{\mathbb{R}^2} \mathcal{U}^k(x-y) F_k(y)\, dy = -\sigma \int_\Gamma \mathcal{U}^k(x-y)(\kappa\nu^k)(y)\, d\Gamma$$

$$= -\sigma \int_{\mathbb{R}} \mathcal{U}^k(x-(s,f(s))(\underbrace{\omega\kappa\nu^k}_{=g_k'})(s)\, ds$$

$$= \sigma \int_{\mathbb{R}} \partial_s \big(\mathcal{U}^k(x-(s,f(s))\big) g_k(s)\, ds$$

and similarly

$$p(x) = -\sigma \int_{\mathbb{R}} \mathcal{P}^k(x-(s,f(s)) g_k'(s)\, ds$$

Plemelj type formulas enable us to compute the traces $v^\pm|_\Gamma$ in terms of certain singular integrals operators $B^0_{n,m}$ (introduced in [M'18, M'19])

The abstract formulation in the case $\mu^+ = \mu^- =: \mu$

In view of $V_n = v^\pm|_\Gamma \cdot \nu$, we arrive at

$$\frac{df}{dt} = \mathcal{F}(f) := -f' v_1|_\Gamma + v_2|_\Gamma$$

with

$$(v_1|_\Gamma, v_2|_\Gamma)^\top = \frac{\sigma}{4\mu}\begin{pmatrix} B^0_{2,2}(f) - B^0_{0,2}(f) & B^0_{3,2}(f) - B^0_{1,2}(f) \\ B^0_{3,2}(f) - B^0_{1,2}(f) & -3B^0_{2,2}(f) - B^0_{0,2}(f) \end{pmatrix}[g]$$
$$+ \frac{\sigma}{4\mu}\begin{pmatrix} -B^0_{3,2}(f) - 3B^0_{1,2}(f) & B^0_{2,2}(f) - B^0_{0,2}(f) \\ -B^0_{2,2}(f) + B^0_{0,2}(f) & -B^0_{3,2}(f) + B^0_{1,2}(f) \end{pmatrix}[f'g]$$

with $g = (g_1, g_2)$ satisfies $g' = \omega\kappa\nu$ and $g_j = g_j(f')$, $j = 1, 2$

Though v has been identified under the assumption $f \in H^3(\mathbb{R})$, the operator $\mathcal{F}$ can be defined on $H^s(\mathbb{R})$ with $s > 3/2$

The abstract formulation in the general $\mu^+ - \mu^- \in \mathbb{R}$

The Stokes flow can be reformulated as the following evolution equation

$$\frac{df}{dt} = \mathcal{F}(f) := -\frac{1}{\mu_+ - \mu_-}\big(- f'\beta_1(f) + \beta_2(f)\big)$$

with $\beta = \beta(f) = (\beta_1(f), \beta_2(f))^\top$ is the unique solution to

$$(1 + 2a_\mu \mathbb{D}(f))[\beta] = 2a_\mu (v_1|_\Gamma, v_2|_\Gamma)^\top$$

and

$$a_\mu = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} \in (-1, 1)$$

and

$$\mathbb{D}(f)[\beta] = \frac{1}{\pi}\begin{pmatrix} B^0_{0,2}(f) & B^0_{1,2}(f) \\ B^0_{1,2}(f) & B^0_{2,2}(f) \end{pmatrix}\begin{pmatrix} f'\beta_1 \\ f'\beta_2 \end{pmatrix} - \frac{1}{\pi}\begin{pmatrix} B^0_{1,2}(f) & B^0_{2,2}(f) \\ B^0_{2,2}(f) & B^0_{3,2}(f) \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

is the double layer potential

Remarks:

- In the case when $\mu_- \neq \mu_+$ we need to solve the equation
$$(1 + 2a_\mu \mathbb{D}(f))[\beta] = 2a_\mu (v_1|_\Gamma, v_2|_\Gamma)^\top$$
- The solution operator $[f \mapsto \beta(f)]$ induces additional nonlinearities and nonlocalities
- We establish the invertibility of $1 + 2a_\mu \mathbb{D}(f)$ by using underlying Rellich formulas for the Stokes problem [Chang & Pahk, '09], but also for the Muskat problem
- In both case we fix $s \in (3/2, 2)$ and show that
  - $[f \mapsto \mathcal{F}(f)] : H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ is smooth
  - The Fréchet derivative $\partial\mathcal{F}(f_0)$ generates an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$ for each $f_0 \in H^s(\mathbb{R})$
- These properties identify the Stokes flow as a parabolic evolution problem [Lunardi, '95]
- The proofs of these two properties rely on the properties of the operators $B^0_{n,m}$

The operators $B^0_{n,m}$

Given Lipschitz continuous maps $a, b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}$ and $n, m \in \mathbb{N}$ and we set

$$B_{n,m}(a)[b_1, \ldots, b_n, h](\xi) := \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{h(\xi - s)}{s} \frac{\prod_{i=1}^{n} \left(\delta_{[\xi,s]} b_i / s\right)}{\left(1 + \left(\delta_{[\xi,s]} a / s\right)^2\right)^m} ds,$$

where $\mathrm{PV} \int_{\mathbb{R}}$ is the principal value integral and $\delta_{[\xi,s]} u := u(\xi) - u(\xi - s)$

- If $n = 0$ and $a = 0$, then $B_{n,m}(0) = H$ is the Hilbert transform
- a is a nonlinear argument and $b_1, \ldots, b_n$ are additional linear arguments
- Given $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous we set

$$B^0_{n,m}(f) = B_{n,m}(f)[f, \ldots, f, \cdot]$$

These are the operators that appear in the abstract formulations

Estimates for the operators $B_{n,m}$

- [M' 18, M '19] There exists a constant $C = C(n, m, \|a'\|_\infty)$ such that
$$\|B_{n,m}(a)[b_1, \ldots, b_n, \cdot]\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq C \prod_{i=1}^n \|b_i'\|_\infty.$$
The proof uses a deep result from harmonic analysis [Murai '86]
- [Abels & M '21] There exists a constant $C = C(s, n, m, \|a\|_{H^s})$, where $s \in (3/2, 2)$, such that
$$\|B_{n,m}(a)[b_1, \ldots, b_n, \cdot]\|_{\mathcal{L}(H^{s-1}(\mathbb{R}))} \leq C \prod_{i=1}^n \|b_i\|_{H^s}$$
for all $b_1, \ldots, b_n \in H^s(\mathbb{R})$
- [M & Prokert '21]
$$[f \mapsto B^0_{n,m}(f)] : H^s(\mathbb{R}) \to \mathcal{L}(H^{s-1}(\mathbb{R})) \quad \text{is smooth}$$
- The smoothness of $[f \mapsto \mathcal{F}(f)] : H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ is a consequence of the latter property

The generator property for $\partial\mathcal{F}(f_0)$

- The first step is to compute the derivative $\partial\mathcal{F}(f_0)$. The leading terms of this operator are expressed again in terms of $B^0_{n,m}(f_0)$ since

$$\partial B^0_{n,2}(f_0)[f][h] = nB_{n,2}(f_0)[f, f_0, \dots f_0, h] - 4B_{n+2,3}(f_0)[f, f_0, \dots, f_0, h]$$

and, since for $s' \in (3/2, s)$ we have [Abels & M, '21]

$$\|B_{n,m}(f_0)[f, f_0, \dots f_0, h] - hB^0_{n-1,m}(f_0)[f']\|_{H^{s-1}} \leq C\|h\|_{H^{s-1}}\|f\|_{H^{s'}},$$

it holds that

$$\partial B^0_{n,2}(f_0)[f][h] = h\big(nB^0_{n-1,2}(f_0)[f'] - 4B^0_{n+1,3}(f_0)[f']\big) + R_n[f, h],$$

where

$$\|R_n[f, h]\|_{H^{s-1}} \leq C\|h\|_{H^{s-1}}\|f\|_{H^{s'}}$$

The generator property for $\partial\mathcal{F}(f_0)$

- If $f_0 = 0$, then $\partial\mathcal{F}(f_0) = \partial\mathcal{F}(0)$ is the Fourier multiplier

$$\partial\mathcal{F}(0) = -\frac{\sigma}{2(\mu_+ + \mu_-)} H \circ \frac{d}{d\xi} = -\frac{\sigma}{2(\mu_+ + \mu_-)} \Big(-\frac{d^2}{d\xi^2} \Big)^{1/2}$$

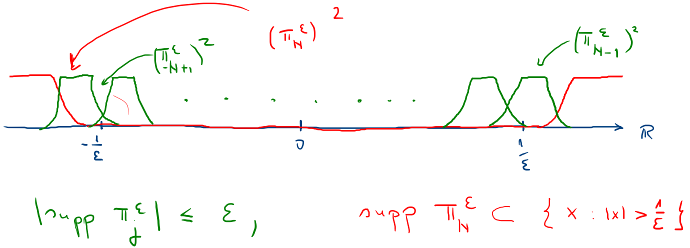
  and generates an analytic semigroup
- If $f_0 \neq 0$, then $\partial\mathcal{F}(f_0)$ is no longer a Fourier multiplier, as it is expressed by means of the singular integral operators $B^0_{n,m}(f_0)$
- $\partial\mathcal{F}(f_0)$ can be locally approximated by

$$\alpha\partial\mathcal{F}(0) + \beta\frac{d}{d\xi}, \qquad \text{with constants } \alpha > 0 \text{ and } \beta \in \mathbb{R}$$

  (this generalizes of the method of "freezing the coefficients" of elliptic differential operators)
- Using a strategy from [Escher '94, Escher & Simonett '95, '97] this leads to the desired generator property
- The analysis is build up on localization results for $B^0_{n,m}(f_0)$

Finite ε-localization family and the localization of $\partial\mathcal{F}(f_0)$

Finite ε−localization family: $\{(\pi_j^\varepsilon, \xi_j^\varepsilon) : -N+1 \le j \le N\} \subset C^\infty(\mathbb{R}) \times \mathbb{R}$



Lemma (Abels & M '21, M & Prokert '20)

Given $3/2 < s' < s < 2$ *and* $\nu > 0$, *it holds for* $\varepsilon \in (0,1)$ *sufficiently small*

$$\left\| \pi_j^\varepsilon B^0_{n,m}(f_0)[h] - \frac{(f_0'(\xi_j^\varepsilon))^n}{[1+(f_0'(\xi_j^\varepsilon))^2]^m} H[\pi_j^\varepsilon h] \right\|_{H^{s-1}} \le \nu \|\pi_j^\varepsilon h\|_{H^{s-1}} + K\|h\|_{H^{s'-1}}$$

for all $|j| \le N-1$ *and* $h \in H^{s-1}(\mathbb{R})$.

The localization of $\partial\mathcal{F}(f_0)$

Theorem (M & Prokert '20, '21)

Let $\nu > 0$ be given and fix $3/2 < s' < s < 2$. Then, there exist $\varepsilon \in (0,1)$, a constant $K = K(\varepsilon)$, and constants

$$\alpha_j^\varepsilon > 0 \qquad \text{and} \qquad \beta_j^\varepsilon \in \mathbb{R}$$

such that

$$\Big\|\pi_j^\varepsilon \partial\mathcal{F}(f_0)[f] - \Big(\alpha_j^\varepsilon \partial\mathcal{F}(0) + \beta_j^\varepsilon \frac{d}{d\xi}\Big)[\pi_j^\varepsilon f]\Big\|_{H^{s-1}} \leq \nu \|\pi_j^\varepsilon f\|_{H^s} + K\|f\|_{H^{s'}}$$

for all $j \in \{-N+1, \dots, N\}$ and $f \in H^s(\mathbb{R})$.

Remarks on the proof of the main result

- The well-posedness and the criterion for global existence follow by using the abstract parabolic theory from [Lunardi '95] (with a slight improvement concerning uniqueness of solutions)
- The parabolic smoothing property uses the translation invariance of the problem (via a parameter trick)

M & Prokert, *Two-phase Stokes flow by capillarity in full 2D space: an approach via hydrodynamic potentials*, Proc. Roy. Soc. Edinburgh Sect. A, 2020, p. 1-31.

M & Prokert, *Two-phase Stokes flow by capillarity in the plane: The case of different viscosities*, arXiv:2102.12814.

Thank you!