The Stochastic Primitive Equations with Transport Noise and Turbulent Pressure

Martin Saal

TU Darmstadt

Based on a work with A. Agresti (IST), M. Hieber (TU Darmstadt) and A. Hussein (TU Kaiserslautern)

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Overview

1. Motivation
2. Stochastic primitive equations with transport noise and turbulent pressure
3. Stratonovich noise for the primitive equations

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Stochastic Navier-Stokes equations for turbulent flows

Stochastic Lagrangian approach

Reynolds 1880:

$$\text{Velocity field} = \underbrace{u}_{\substack{\text{Slow oscillating part} \\ \text{(deterministic)}}} + \underbrace{\phi\dot{\beta}}_{\substack{\text{Fast oscillating part} \\ \text{(random)}}}$$

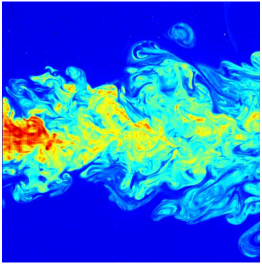

Image source: https://en.wikipedia.org/wiki/Turbulence

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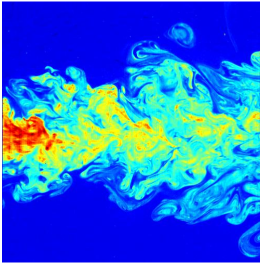

Kraichnan's turbulence theory (1968)

1. Statistic modeling: $\phi \in C^\alpha$ for some $\alpha > 0$;
2. Newton's law yields

$$du - \Delta u\, dt = \big(- \nabla P - (u \cdot \nabla) u \big) dt + \sum_{n \geq 1} \underbrace{(\phi_n \cdot \nabla) u}_{\substack{\text{Stochastic}\\ \text{transport}}} \, d\beta_t^n. \qquad (1)$$

Eq. (1) is called Navier-Stokes equations for turbulent flows.

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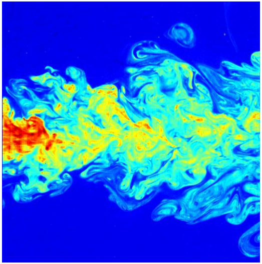

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1. Statistic modeling: $\phi \in H^{\eta,\xi}$ for some $\eta > d/\xi$ and $\xi \in [2,\infty)$;
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$$du - \Delta u\, dt = \big(-\nabla P - (u\cdot\nabla)u\big)dt + \textstyle\sum_{n\geq 1} \underbrace{(\phi_n\cdot\nabla)u}_{\substack{\text{Stochastic}\\ \text{transport}}}\, d\beta_t^n. \qquad (1)$$

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Transport noise as a model for small-scales

Separation of scales

Suppose that $u = u_L + u_S$ where L stands for "Large" and S for "Small" scale and

$$\partial_t u_L - \Delta u_L = -\nabla P_L - ((u_L + u_S) \cdot \nabla) u_L,$$
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In a turbulent regime one can model u_S as an approximation of white noise, so that

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Transport noise preserves scaling

Solutions to the Navier-Stokes equations are invariant under the scaling

$$u(t,x) \mapsto u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^{1/2} u(\lambda t, \lambda^{1/2} x) \quad \text{where} \quad \lambda > 0. \tag{2}$$

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has the same scaling of

$$\int_0^{t/\lambda} (u_\lambda(s,x) \cdot \nabla) u_\lambda(s,x)\, ds = \lambda^{1/2} \int_0^t (u(s,\lambda x) \cdot \nabla) u(s,\lambda x)\, ds.$$

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Stochastic transport perturbation of the NS equations preserves its natural scaling!

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For $\varepsilon > 0$ let $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (-\varepsilon, 0)$. Consider the following anisotropic stochastic Navier-Stokes equations on $\mathcal{O}_\varepsilon$:

$$\begin{aligned} du - \big(\Delta_{\mathrm{H}} u + \varepsilon^2 \partial_3^2 u\big)\, dt &= \big[-\nabla P + (u \cdot \nabla) u \big]\, dt \\ &\quad + \sum_{n \geq 1} \Big[(\phi_{n,\mathrm{H}} \cdot \nabla_{\mathrm{H}}) u + \varepsilon \phi_n^3 \partial_3 u \Big]\, d\beta_t^n \end{aligned} \tag{3}$$

here H stands for horizontal component, i.e. $\Delta_{\mathrm{H}} = \partial_1^2 + \partial_2^2$, $\nabla_{\mathrm{H}} = (\partial_1, \partial_2)$ and $\phi_{n,\mathrm{H}} = (\phi_n^1, \phi_n^2)$ for $\phi_n = (\phi_n^1, \phi_n^2, \phi_n^3)$.

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The primitive equations are the limit of (3) as $\varepsilon \downarrow 0$.

Decompose $u = (v, w)$ where $v : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}^2$, and $w : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}$.

Hydrostatic approximation and primitive equations II

Decompose $u = (v, w)$ where $v : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}^2$, and $w : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}$.
The following rescaling yields unknown on $\mathcal{O} \overset{\mathrm{def}}{=} \mathcal{O}_1 = \mathbb{T}^2 \times (-1, 0)$:

$$P_\varepsilon(x_{\mathrm{H}}, x_3) = P(x_{\mathrm{H}}, \varepsilon x_3), \quad v_\varepsilon(x_{\mathrm{H}}, x_3) = v(x_{\mathrm{H}}, \varepsilon x_3), \quad w_\varepsilon(x_{\mathrm{H}}, x_3) = \varepsilon^{-1} w(x_{\mathrm{H}}, \varepsilon x_3).$$

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Thus $v_\varepsilon, w_\varepsilon$ satisfy

$$\begin{aligned} d v_\varepsilon &= \left[\Delta v_\varepsilon - \nabla P_\varepsilon + (u_\varepsilon \cdot \nabla) v_\varepsilon\right] dt + \sum_{n \geq 1} (\phi_n \cdot \nabla) v_\varepsilon \, d\beta_t^n, \\ d(\varepsilon^2 w_\varepsilon) &= \left[\varepsilon^2 (\Delta w_\varepsilon - (u_\varepsilon \cdot \nabla) w_\varepsilon) - \partial_3 P_\varepsilon\right] dt + \sum_{n \geq 1} \varepsilon^2 (\phi_n \cdot \nabla) w_\varepsilon \, d\beta_t^n. \end{aligned} \tag{4}$$

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Hydrostatic approximation

The formal limit as $\varepsilon \downarrow 0$ in (4) yields the Hydrostatic approximation by replacing (4) by

$$\partial_3 P_\varepsilon = 0 \quad \Longrightarrow \quad P_\varepsilon(x_{\mathrm{H}}, x_3) = p_\varepsilon(x_{\mathrm{H}}).$$

The unknown p_ε is usually called surface pressure.

Furukawa, Giga, Hieber, Hussein, Kashiwabara, & Wrona ('20). Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier–Stokes equations. *Nonlinearity, 33(12), 6502.*

Differences between Navier-Stokes and Primitive equations

Navier-Stokes	**Primitive equations**
Evolution equation for w	Constraint for the pressure
Constraint for the velocity	"‘evolution"’ equation for w
All directions are equivalent	Anisotropic behavior
Full pressure $p(x, y, z)$	Surface pressure $p(x, y)$
$\mathrm{div} u = 0$ (full divergence)	$\mathrm{div}_H \overline{v} = 0$ (horizontally divergence)
None of the velocity direction can be substituted	Vertical velocity is known, $w(\cdot, z) = -\mathrm{div}_H \int_{-h}^{z} v(\cdot, \xi) d\xi$

Primitive equations

The hydrostatic approximation yields the following system on $\mathcal{O} = \mathbb{T}^2 \times (-1, 0)$:

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3. The divergence free condition uniquely determines p (up to a constant).

Hydrostatic Helmholtz projection

For $f \in L^2(\mathcal{O};\mathbb{R}^2)$, let $\Psi_f \in H^1(\mathbb{T}^2)$ be such that

$$\Delta_{\mathrm{H}}\Psi_f = \mathrm{div}_{\mathrm{H}}\left[\int_{-1}^{0} f(\cdot,\zeta)\,d\zeta\right] \quad \text{and} \quad \int_{\mathbb{T}^2}\Psi_f\,dx_{\mathrm{H}} = 0.$$

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- Orthogonality property: $\mathbb{P}[\nabla_{\mathrm{H}}p] = 0$ for all $p \in H^1(\mathbb{T}^2)$.

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Reformulation

We are looking for a process $v : \mathbb{R}_+ \times \Omega \times \mathcal{O} \to \mathbb{R}^2$ such that $v(0) = v_0$ and

$$dv = \Big[\Delta v - \mathbb{P}[(v\cdot\nabla_{\mathrm{H}})v - w(v)\partial_3 v]\Big]\,dt + \sum_{n\geq 1}\mathbb{P}[(\phi_n\cdot\nabla)v]\,d\beta_t^n,$$

where $w(v) = -\int_{-1}^{x_3}\mathrm{div}_{\mathrm{H}}\,v(\cdot,\zeta)\,d\zeta$ and satisfying

$$\partial_3 v(\cdot,-1) = \partial_3 v(\cdot,0) = 0 \quad \text{on } \mathbb{T}^2.$$

Theorem – Agresti, Hieber, Hussein and S '21

Let $v_0 \in H^1(\mathcal{O};\mathbb{R}^2)$ be such that $\int_{-1}^{0} \mathrm{div}_{\mathrm{H}} v_0(\cdot,\zeta)\, d\zeta = 0$ a.s. Under suitable assumptions on $(\phi_n)_{n\geq 1}$, there exists a unique local and maximal solution

$$v \in L^2_{\mathrm{loc}}([0,\tau); H^2(\mathcal{O};\mathbb{R}^2)) \cap C([0,\tau); H^1(\mathcal{O};\mathbb{R}^2)), \quad \tau > 0 \text{ a.s.}$$

to

$$\begin{cases} dv = \mathbb{P}[\Delta v - (v\cdot\nabla_{\mathrm{H}})v - w(v)\partial_3 v]\, dt + \sum_{n\geq 1} \mathbb{P}[(\phi_n\cdot\nabla)v]\, d\beta_t^n, & \text{on } \mathcal{O}, \\ v(0) = v_0, \qquad \partial_3 v(\cdot,-1) = \partial_3 v(\cdot,0) = 0, & \text{on } \mathbb{T}^2. \end{cases}$$

The stochastic primitive equations with transport noise and turbulent pressure. *arXiv preprint, arXiv:2109.09561*.

Local and global existence for primitive equations

Theorem – Agresti, Hieber, Hussein and S '21

Let $v_0 \in H^1(\mathcal{O};\mathbb{R}^2)$ be such that $\int_{-1}^{0} \operatorname{div}_{\mathrm{H}} v_0(\cdot,\zeta)\, d\zeta = 0$ a.s. Under suitable assumptions on $(\phi_n)_{n\geq 1}$, there exists a unique local and maximal solution

$$v \in L^2_{\mathrm{loc}}([0,\tau); H^2(\mathcal{O};\mathbb{R}^2)) \cap C([0,\tau); H^1(\mathcal{O};\mathbb{R}^2)), \quad \tau > 0 \text{ a.s.}$$

to

$$\begin{cases} dv = \mathbb{P}[\Delta v - (v\cdot\nabla_{\mathrm{H}})v - w(v)\partial_3 v]\, dt + \sum_{n\geq 1} \mathbb{P}[(\phi_n\cdot\nabla)v]\, d\beta_t^n, & \text{on } \mathcal{O}, \\ v(0) = v_0, \qquad \partial_3 v(\cdot,-1) = \partial_3 v(\cdot,0) = 0, & \text{on } \mathbb{T}^2. \end{cases}$$

If ϕ_n^1, ϕ_n^2 are independent of x_3 ($\phi_n = (\phi_n^j)_{j=1}^3$), then v solution is global, i.e. $\tau = \infty$ a.s.

The stochastic primitive equations with transport noise and turbulent pressure. *arXiv preprint, arXiv:2109.09561*.

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Some comments

1. Regularity of $\phi_n = (\phi_n^j)_{n\geq 1}$. For $j \in \{1,2,3\}$ and some $\delta > 0$

$$(\phi_n^j)_{n\geq 1} \in H^{1,3+\delta}(\mathcal{O}; \ell^2);$$

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3. The turbulent pressure is the "random" component of the pressure and arises from
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4. Physical relevant noise. Stratonovich noise leads to variable viscosity:

$$\Delta v \rightsquigarrow \mathrm{div}(a_\phi \cdot \nabla v), \quad \text{where} \quad a_\phi^{i,j} \stackrel{\mathrm{def}}{=} \delta^{i,j} + \frac{1}{2}\sum_{n\geq 1} \phi_n^i \phi_n^j.$$

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- The semigroup approach is not (directly) applicable;
- The estimate we need is typically called stochastic maximal L^p-regularity.

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2. Blow-up criterium for maximal solutions;

2 Gluing together local solutions one obtains a maximal local solution up to time τ. Arguing by contradiction one gets[1]

$$\mathsf{P}\Big(\tau < T, \sup_{t\in[0,\tau)} \|v(t)\|_{H^1}^2 + \int_0^\tau \|v(t)\|_{H^2}^2 \, dt < \infty\Big) = 0, \quad T \in (0,\infty).$$

[1] Agresti & Veraar ('20). Nonlinear parabolic stochastic evolution equations in critical spaces Part II. Blow-up criteria and instantaneous regularization. To appear in *Journal of Evolution Equations*.

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The proof splits into three parts:

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3. Global existence by combining energy estimates and blow-up criteria.

3. Assume that the following energy estimate holds: For $T > 0$

$$\mathrm{E}\Big[\sup_{t\in[0,\tau)} \|v(t)\|_{H^1}^2\Big] + \mathrm{E}\int_0^{\tau} \|v(t)\|_{H^2}^2\,dt \leq C_T(1+\mathrm{E}\|v_0\|_{H^1}^2)$$

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- $\mathrm{P}(\tau < T) \overset{\text{Energy estimate}}{=} \mathrm{P}\Big(\tau < T, \sup_{t\in[0,\tau)} \|v(t)\|_{H^1} + \int_0^\tau \|v(t)\|_{H^2}^2\,dt < \infty\Big) \overset{\text{Blow-up criterium}}{=} 0;$

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Stochastic maximal L^2-regularity

Let $T \in (0, \infty)$. Under suitable assumption, for each

$$f \in L^2((0,T) \times \Omega; \mathbb{L}^2), \quad \text{and} \quad g = (g_n)_{n\geq 1} \in L^2((0,T) \times \Omega; \mathbb{H}^1(\ell^2))$$

the unique solution v to

$$\begin{cases} dv = (\Delta v + f)\,dt + \sum_{n\geq 1} \Big(\mathbb{P}[(\phi_n \cdot \nabla) v] + g_n \Big)\, d\beta_t^n, & \text{on } \mathcal{O}, \\ v(0) = 0, \qquad \partial_3 v(\cdot, -1) = \partial_3 v(\cdot, 0) = 0 & \text{on } \mathbb{T}^2. \end{cases}$$

satisfies

$$\|v\|_{L^2((0,T)\times\Omega;H^2)} \lesssim \|f\|_{L^2((0,T)\times\Omega;L^2)} + \|g\|_{L^2((0,T)\times\Omega;H^1(\ell^2))}.$$

Here $\mathcal{O} = \mathbb{T}^2 \times (-1, 0)$,

$$\mathbb{L}^2 \stackrel{\text{def}}{=} \mathbb{P}(L^2(\mathcal{O}; \mathbb{R}^2)) \quad \text{and} \quad \mathbb{H}^1(\ell^2) = H^1(\mathcal{O}; \ell^2(\mathbb{N}; \mathbb{R}^2)) \cap \mathbb{L}^2(\mathcal{O}; \ell^2(\mathbb{N}; \mathbb{R}^2)).$$

1. **Method of continuity.** For $\lambda \in [0,1]$ consider

$$\begin{cases} dv = (\Delta v + f)\,dt + \sum_{n\geq 1}\Big(\lambda \mathbb{P}[(\phi_n \cdot \nabla)v] + g_n\Big)\,d\beta_t^n, & \text{on } \mathcal{O}, \\ v(0) = 0, & \partial_3 v(\cdot,-1) = \partial_3 v(\cdot,0) = 0 \ \text{ on } \mathbb{T}^2. \end{cases}$$

and prove the a-priori estimate with C independent of λ

$$\|v\|_{L^2((0,T)\times\Omega;H^2)} \leq C\Big(\|f\|_{L^2((0,T)\times\Omega;L^2)} + \|g\|_{L^2((0,T)\times\Omega;H^1(\ell^2))}\Big).$$

Sketch of the proof

1. **Method of continuity.** Prove

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2. **Apply the Itô's formula to $v \mapsto \|\nabla v\|_{L^2}^2$.** Integrating by parts, on the LHS one has $2\mathsf{E}\int_0^T\int_{\mathcal{O}}|\Delta v|^2\,dxds$ and on the RHS

$$\sum_{k=1}^{3}\mathsf{E}\int_0^T \|\lambda\mathbb{P}[(\phi_n\cdot\nabla)\partial_k v]\|_{L^2}^2\,ds \leq \sum_{k=1}^{3}\sum_{n\geq 1}\mathsf{E}\int_{\mathcal{O}}\Big|\sum_{j=1}^{3}\phi_n^j\partial_{j,k}^2 v\Big|^2\,dxds$$

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(Parabolicity) $$\leq \nu\sum_{j,k=1}^{3}\mathrm{E}\int_0^T\int_{\mathcal{O}}|\partial_{j,k}^2 v|^2\, dxds$$

parabolicity means $\sum_{n\geq 1}\big(\sum_{j=1}^3 \phi_n^j\xi_j\big)^2 \leq \nu|\xi|^2$ for some $\nu\in(0,2)$.

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(Kadlec's formula) $$\le \nu'\mathsf{E}\int_0^T\int_{\mathcal{O}} |\Delta v|^2\, dxds + c_{\nu,\nu'}\mathsf{E}\int_0^T\int_{\mathcal{O}} |v|^2\, dxds$$

where $\nu' \in (\nu, 2)$ and parabolicity means $\sum_{n\ge1}\big(\sum_{j=1}^3 \phi_n^j\xi_j\big)^2 \le \nu|\xi|^2$ for some $\nu \in (0,2)$.

Energy estimate

In addition to the assumptions of the local existence, suppose ϕ_n^1, ϕ_n^2 are independent of x_3. Then for each $T \in (0, \infty)$ there exist stopping times $(\mu_k)_{k \geq 1}$ such that

$$\lim_{k \to \infty} \mathrm{P}(\mu_k = \tau \wedge T) = 1, \quad \mathrm{E}\Big[\sup_{t \in [0, \mu_k)} \|v(t)\|_{H^1} \Big] + \mathrm{E} \int_0^{\mu_k} \|v(s)\|_{H^1}^2 \, ds \lesssim_{k,T} 1 + \mathrm{E}\|v_0\|_{H^1}^2 .$$

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Key observation: Split $v = \overline{v} + \widetilde{v}$ where $\overline{v} \stackrel{\text{def}}{=} \int_{-1}^0 v(\cdot,\zeta)\, d\zeta,$ and $\widetilde{v} \stackrel{\text{def}}{=} v - \overline{v}$. Moreover $\overline{v}$ and $\widetilde{v}$ solves a 2D Navier-Stokes equations and 3D heat equation, respectively.

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- Estimates for $\|\overline{v}\|_{H^1}^2$ and $\|\overline{v}\|_{H^1}^2$ are available in terms of L^4-norms of $\widetilde{v}$.

1 Motivation

2 Stochastic primitive equations with transport noise and turbulent pressure

3 Stratonovich noise for the primitive equations

There are different ways to define the stochastic integral:

Stratonovich noise

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Itô-integral

$$\int_0^t g(s)d\beta = \lim_{n\to\infty} \sum_{i=1}^{n} g(s_{i-1})(\beta_{s_i} - \beta_{s_{i-1}})$$

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Advantage: Itô-isometry, $\mathrm{E}\left[\left(\int_0^t g(s)d\beta\right)^2\right] = \mathrm{E}\left[\int_0^t g(s)^2 ds\right]$.

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Advantage: Calculus similar to deterministic one.

1. The two integrals differ by a correction term:

$$\mathbb{P}[(\phi_n \cdot \nabla) v] \underbrace{\circ d\beta_t^n}_{\substack{\text{Stratonovich}\\ \text{noise}}} = \mathbb{P}[(\phi_n \cdot \nabla) v] \underbrace{d\beta_t^n}_{\substack{\text{Itô}\\ \text{noise}}} + \underbrace{\frac{1}{2}\mathbb{P}\Big[(\phi_n \cdot \nabla)\big(\mathbb{P}[(\phi_n \cdot \nabla) v]\big)\Big]\, dt}_{\text{Correction term}}.$$

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2. Formally[2], one has by rewriting the correction term

$$\mathbb{P}[(\phi_n\cdot\nabla)v]\circ d\beta_t^n = \mathbb{P}[(\phi_n\cdot\nabla)v]\,d\beta_t^n + \mathbb{P}\big[L_\phi v + P_\phi v\big]\,dt,$$

where

$$L_\phi v := \underbrace{\Delta v}_{\substack{\text{disspation}\\ \text{by noise}}} + \underbrace{\frac{1}{2}\sum_{i,j=1}^{3}\phi_n^i(x)\phi_n^j(x)\partial_{i,j}^2 v}_{\text{variabel disspation}} + \frac{1}{2}\sum_{i,j=1}^{3}\sum_{n\geq 1}(\partial_i\phi_n^j)\phi_n^i\partial_j v$$

and P_ϕ is a lower order term.

[2]see Flandoli ('21). Stochastic Partial Differential Equations in Fluid Mechanics 2021. https://www.waseda.jp/inst/sgu/news-en/2021/03/08/8586/.

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[3] Giga, Gries, Hieber, Hussein & Kashiwabara ('20). Analyticity of solutions to the primitive equations. Mathematische Nachrichten, 293(2), 284-304.

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[4] Flandoli & Luo ('21). High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. Probability Theory and Related Fields, 180(1), 309-363.

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Thank you for your attention!