Modeling of fluid flow in a flexible vessel with elastic walls

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Blood circulatory system

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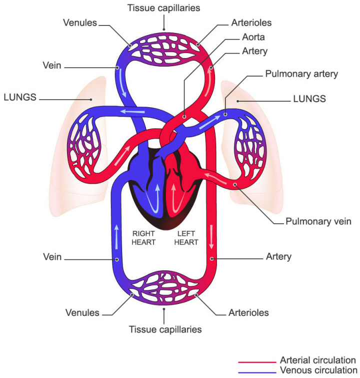

Figure: Circulatory system

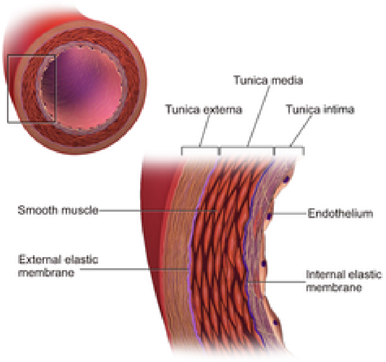

Figure: Layer's structure of the wall

Plan of the talk

1. $3D$ model.
2. Model with $2D$ elastic wall.
3. Periodic solutions.
4. Discussion
5. Further results.

$3D$ Mathematical model of vessel with layered wall

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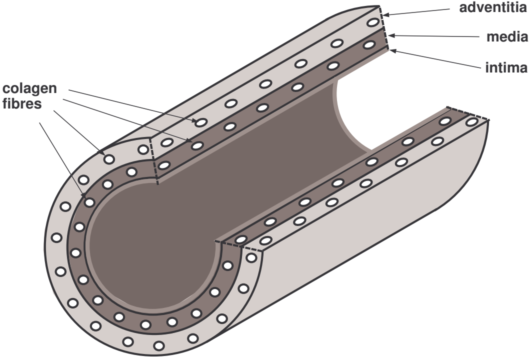

Figure: The wall of blood vessel consisting of three layers reenforced by collagen fibres.

Cross-section

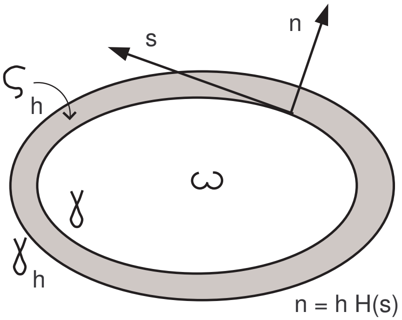

Figure: The cross-section of the blood vessel

Let Ω be a bounded, simple connected domain in $\mathbb{R}^2$ with boundary $\gamma = \partial\Omega$ and let us introduce the cylinder (the lumen of the vessel)

$$\mathcal{C} = \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : \; y = (y_1, y_2) \in \Omega, \; z \in \mathbb{R}\}. \tag{1}$$

The curve γ is parameterised as $(y_1, y_2) = (\zeta_1(s), \zeta_2(s))$, where s is the arc length along γ measured counterclockwise from a certain point. The length of the countour γ is denoted by $|\gamma|$ and its curvature by

$$\kappa = \kappa(s) = \zeta_1''(s)\zeta_2'(s) - \zeta_2''(s)\zeta_1'(s).$$

In a neighborhood of γ, we introduce the natural curvilinear orthogonal coordinates system (n, s), where n is the oriented distance to γ ($n > 0$ outside Ω).

The boundary of the cylinder $\mathcal{C}$ is denoted by Γ, i.e.

$$\Gamma = \{x = (y, z) : \, y \in \gamma, z \in \mathbb{R}\}. \tag{2}$$

Denote by ς_h the domain between γ and γ_h. Then the wall of the cylinder $\mathcal{C}$ is

$$\Sigma_h = \varsigma_h \times \mathbb{R}.$$

An appropriate rescaling makes parameters and the coordinates dimensionless.

The flow in the vessel is described by the velocity vector $\mathbf{v} = (v_1, v_2, v_3)$ and by the pressure p which are subject to the Navier–Stokes equations:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p \ \text{ and } \ \nabla \cdot \mathbf{v} = 0 \ \text{ in } \mathcal{C}. \tag{3}$$

Here $\nu = \mu/\rho_b$ is the kinematic viscosity, ρ_b is the density of the fluid and μ is the dynamic viscosity.

The stress state of the linear elastic wall is described by the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ and by the stress tensor $\sigma = \{\sigma_{jk}\}_{j,k=1}^3$ subject to the nonstationary elasticity equations

$$\frac{\partial \sigma_{j1}}{\partial x_1} + \frac{\partial \sigma_{j2}}{\partial x_2} + \frac{\partial \sigma_{j3}}{\partial x_3} = \rho \frac{\partial^2 u_j}{\partial t^2} \quad \text{in } \Sigma_h, \quad \text{j=1,2,3}, \tag{4}$$

and Hooke's law

$$\sigma_{jk} = \sum_{p,q=1}^{3} A_{jk}^{pq} \varepsilon_{pq}, \quad j,k = 1,2,3, \quad \varepsilon_{pq} = \frac{1}{2}\Big(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p} \Big). \tag{5}$$

Here ρ is the mass density, $\varepsilon = \{\varepsilon_{jk}\}_{j,k=1}^3$ is the strain tensor, while the rigidity tensor $\mathbf{A} = \{A_{jk}^{pq}\}$ (also called Hooke's tensor), consists of the moduli of elasticity of the wall material and has the standard symmetry and positivity properties:

$$A^{pq}_{jk} = A^{jk}_{pq} = A^{kj}_{pq},$$

$$\sum_{j,k,p,q=1}^{3} A^{pq}_{jk}\xi_{jk}\xi_{pq} \geq C_A \sum_{j,k=1}^{3} |\xi_{jk}|^2.$$

Here $C_A > 0$ and $\{\xi_{jk}\}$ is an arbitrary symmetric 3×3-matrix.

On the interior surface $\Gamma = \Gamma_{\text{int}}$, it is natural to impose two conditions: the kinematic no-slip boundary condition

$$\mathbf{v} = \partial_t \mathbf{u} \quad \text{on } \Gamma, \tag{6}$$

and the dynamic condition (the hydrodynamic force equals the normal stress vector):

$$\sigma_\Gamma := \sigma \cdot \mathbf{n} = \rho_b \mathbf{F}, \tag{7}$$

where

$$\mathbf{F} = -p\mathbf{n} + 2\nu\varepsilon(\mathbf{v})\mathbf{n}$$

or $\mathbf{F} = (F_n, F_z, F_s)$,

$$F_n = -p + \nu \frac{\partial v_n}{\partial n}, \quad F_s = \frac{\nu}{2}\Big(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \kappa u_s\Big), \quad F_z = \frac{\nu}{2}\Big(\frac{\partial v_n}{\partial z} + \frac{\partial v_z}{\partial n}\Big),$$

where v_n and v_s are the velocity components in the direction of the normal $\mathbf{n}$ and the tangent $\mathbf{s}$, respectively, whereas v_z is the longitudinal velocity component $(z = x_3)$; finally, $\kappa(s)$ is the curvature of the contour γ at the point s.

On the outer boundary $\Gamma_{\text{out}} = \gamma_h \times \mathbb{R}$, we have a balance of forces exerted by the surrounding material, external forces and traction. Hence, again with the same small parameter h, we get

$$\sigma\mathbf{n} + hK\mathbf{u} = h\mathbf{f} \ \text{ on } \Gamma_{\text{out}},$$

where hK is the tensor corresponding to the elastic response of the surrounding supporting material so that $Ku = k(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ for some given constant k and $\mathbf{f}$ is the normalised force exerted on the pipe by external factors. In most cases, $\mathbf{f} = 0$ as the effect of external factors are negligible compared to the forces exerted by the surrounding material.

A laminate wall with layers of piecewise constant thickness

We assume that

$$\rho = \frac{1}{h}\rho\Big(\frac{n}{h}, s, z\Big) \quad \text{and} \quad \mathbf{A} = \frac{1}{h}\mathbf{A}\Big(\frac{n}{h}, s, z\Big)$$

and ρ and $\mathbf{A}$ are defined as follows. Let $h_1, \ldots, h_N$ be the given numbers that satisfying

$$h_1, \ldots, h_N > 0, \quad h_1 + \cdots + h_N = h, \quad a_0 = 0, a_j = a_{j-1} + h_j,$$

for $j = 1, \ldots, N$. Then

$$\rho(\zeta, s, z) = \rho^j(s, z), \quad \mathbf{A}(\zeta, s, z) = \mathbf{A}^j(s, z) \quad \text{for } \zeta \in (a_{j-1}/h, a_j/h),$$

where ρ^j and $\mathbf{A}^j$ do not depend on ζ.

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Mathematical model of vessel with $2D$ elastic wall

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The flow in the vessel is described by the velocity vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and by the pressure p which are subject to the non-stationary Stokes equations:

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = 0 \;\text{ and }\; \nabla \cdot \mathbf{v} = 0 \;\text{ in } \mathcal{C} \times \mathbb{R} \ni (x,t). \tag{8}$$

The elastic properties of the 2D boundary are described by the displacement vector $\mathbf{u}$ defined on Γ. If we use the curve-linear coordinates (s, z) on Γ and write the vector $\mathbf{u}$ in the basis $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{z}$, where $\mathbf{n}$ is the outward unit normal vector, $\boldsymbol{\tau}$ is the tangent vector to the curve γ and $\mathbf{z}$ is the direction of z axis, then the balance equation has the following form:

$$\begin{aligned} &D(\kappa, -\partial_s, -\partial_z)^T Q(s) D(\kappa, \partial_s, \partial_z)\mathbf{u} \\ &+\rho(s)\partial_t^2\mathbf{u} + K(s)\mathbf{u} + \rho_b(s)\mathcal{F} = 0 \ \text{ in } \Gamma \times \mathbb{R}, \end{aligned} \tag{9}$$

where $\rho(s)$ is the average density of the vessel wall, A^T stands for the transpose of a matrix A, $D(\kappa, \partial_s, \partial_z) = D_0\partial_z + D_1(\partial_s)$, where

$$D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \; D_1(\partial_s) = \begin{pmatrix} \kappa & \partial_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}\partial_s \end{pmatrix} \qquad (10)$$

and

$$K(s)\mathbf{u} = (k(s)\mathbf{u}_1, 0, 0). \qquad (11)$$

Here $k(s)$ is a scalar function, Q is a 3×3 symmetric positive definite matrix of homogenized elastic moduli and the displacement vector $\mathbf{u}$ is written in the curve-linear coordinates (s, z) in the basis $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{z}$.

Furthermore $\mathcal{F} = (\mathcal{F}_n, \mathcal{F}_z, \mathcal{F}_s)$ is the hydrodynamical force given by

$$\mathcal{F}_n = -p + 2\nu \frac{\partial v_n}{\partial n}, \quad \mathcal{F}_s = \nu\Big(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \kappa v_s\Big), \quad \mathcal{F}_z = \nu\Big(\frac{\partial v_n}{\partial z} + \frac{\partial v_z}{\partial n}\Big),$$

where v_n and v_s are the velocity components in the direction of the normal $\mathbf{n}$ and the tangent $\boldsymbol{\tau}$, respectively, whereas v_z is the longitudinal velocity component. The functions ρ, k and the elements of the matrix Q are bounded measurable functions satisfying

$$\rho(s) \geq \rho_0 > 0 \quad \text{and} \quad k(s) \geq k_0 > 0,$$

where k is the function in (11). The elements of the matrix Q are assumed to be Lipschitz continuous and $\langle Q\xi, \xi\rangle \geq q_0|\xi|^2$ for all $\xi \in \mathbb{R}^3$ with $q_0 > 0$, where $\langle\cdot,\cdot\rangle$ is the cartesian inner product in $\mathbb{R}^3$.

We note that

$$D(\kappa, \partial_s, \partial_z)\mathbf{u}^T = (\kappa \mathbf{u}_1 + \partial_s \mathbf{u}_2, \partial_z \mathbf{u}_3, \frac{1}{\sqrt{2}}(\partial_z \mathbf{u}_2 + \partial_s \mathbf{u}_3))^T$$

and one can easily see that

$$\kappa \mathbf{u}_1 + \partial_s \mathbf{u}_2 = \boldsymbol{\varepsilon}_{ss}(\mathbf{u}), ;\ \partial_z \mathbf{u}_3 = \boldsymbol{\varepsilon}_{zz}(\mathbf{u}) \text{ and } \partial_z \mathbf{u}_2 + \partial_s \mathbf{u}_3 = 2\boldsymbol{\varepsilon}_{sz}(\mathbf{u})$$

on Γ. Here $\boldsymbol{\varepsilon}_{ss}(\mathbf{u})$, $\boldsymbol{\varepsilon}_{zz}(\mathbf{u})$ and $\boldsymbol{\varepsilon}_{sz}(\mathbf{u})$ are components of the deformation tensor in the basis $\{\mathbf{n}, \boldsymbol{\tau}, \mathbf{z}\}$. In what follows we will write the displacement vector as $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where $\mathbf{u}_1 = \mathbf{u}_n$, $\mathbf{u}_2 = \mathbf{u}_s$ and $\mathbf{u}_3 = \mathbf{u}_z$. For the velocity $\mathbf{v}$ we will use indexes $1, 2$ and 3 for the components of $\mathbf{v}$ in y_1, y_2 and z directions respectively. Furthermore the vector functions $\mathbf{v}$ and $\mathbf{u}$ are connected on the boundary by the relation

$$\mathbf{v} = \partial_t \mathbf{u} \text{ on } \Gamma \times \mathbb{R}. \tag{12}$$

In our last paper

V Kozlov, S Nazarov, G Zavorokhin, Modeling of fluid flow in a flexible vessel with elastic walls, Journal of Mathematical Fluid Mechanics 23 (3), 1-29, 2021.

we study some properties of the above problem with two-dimensional elastic wall and in the second part of my talk I will present these results.
This problem appears when we deal with a flow in a pipe surrounded by a thin layered elastic wall which separates the flow from the muscle tissue. Since we have in mind an application to the blood flow in the blood circulatory system, we are interested in periodic in time solutions. Our goal is to describe all periodic solutions to the problem, which are bounded in $\mathbb{R} \times \mathcal{C} \ni (t, x)$.

It is reasonable to compare property of solutions to this problem with similar properties of solutions to the Stokes system supplied with the no-slip boundary condition

$$\mathbf{v} = 0 \ \text{ on } \Gamma. \tag{13}$$

We start by the following result about the Dirichlet problem which is possibly known.

Let the boundary γ be Lipschitz and $\Lambda > 0$. There exists $\delta > 0$ such that if $(\mathbf{v}, p)$ are Λ-periodic in time functions satisfying (8), (13) and may admit a certain exponential growth at infinity

$$\max_{0\le t\le\Lambda} \int_{\mathcal{C}} e^{-2\delta|z|}(|\nabla \mathbf{v}(x,t)|^2 + |\nabla \partial_z \mathbf{v}|^2 + |p(x,t)|^2)dx < \infty. \tag{14}$$

Then

$$p = zp_*(t) + p_0(t), \quad \mathbf{v}(x,t) = (0, 0, \mathbf{v}_3(y,t)), \tag{15}$$

where p and $\mathbf{v}_3$ are Λ-periodic functions in t which satisfy the problem

$$\begin{aligned} &\partial_t \mathbf{v}_3 - \nu \Delta_y \mathbf{v}_3 + p_*(t) = 0 \;\text{ in } \Omega \times \mathbb{R} \\ &\mathbf{v} = 0 \;\text{ on } \gamma \times \mathbb{R}. \end{aligned} \tag{16}$$

Let the boundary γ be $C^{1,1}$ and $\Lambda > 0$. Let also $(\mathbf{v}, p, \mathbf{u})$ be a Λ-periodic with respect to t solution to the problem (8)–(12) admitting an arbitrary power growth at infinity

$$\max_{0\le t\le\Lambda}\Big(\int_{\mathcal{C}}(1+|z|)^{-N}(|\mathbf{v}|^2+|\nabla_x\mathbf{v}|^2+|\nabla_x\partial_z\mathbf{v}|^2+|p|^2)dx \tag{17}$$
$$+\int_{\Gamma}(1+|z|)^{-N}(|u|^2+\sum_{k=2}^{3}(|\nabla_{sz}u_k|^2+|\nabla_{sz}\partial_z u_k|^2))dsdz\Big)<\infty$$

for a certain $N > 0$. Then

$$p = zp_0 + p_1, \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \quad \mathbf{v}_3 = p_0\mathbf{v}_*(y), \tag{18}$$

where p_0 and p_1 are constants and $\mathbf{v}_*$ is the Poiseuille profile, i.e.

$$\nu\Delta_y\mathbf{v}_* = 1 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{v}_* = 0 \ \text{ on } \gamma. \tag{19}$$

The boundary displacement vector $\mathbf{u} = \mathbf{u}(s, z)$ satisfies the equation

$$D(\kappa(s), -\partial_s, -\partial_z)^T Q(s) D(\kappa(s), \partial_s, \partial_z)\mathbf{u} + K\mathbf{u} = \sigma(p, 0, p_0 \nu \partial_n \mathbf{v}_3 | \gamma)^T.$$

If the elements Q_{21} and Q_{31} vanish then the function $\mathbf{u}$ is a polynomial of second degree in z:

$$\mathbf{u}(s, z) = (0, \alpha, \beta)^T z^2 + \mathbf{u}^{(1)}(s) z + \mathbf{u}^{(2)}(s),$$

where α and β are constants.

Some ideas of proofs

Due to Λ-periodicity of our solution we can represent it in the form

$$\mathbf{v}(x,t) = \sum_{k=-\infty}^{\infty} \mathbf{V}_k(x) e^{2\pi kit/\Lambda}, \quad p(x,t) = \sum_{k=-\infty}^{\infty} \mathbf{P}_k(x) e^{2\pi kit/\Lambda},$$

where

$$\mathbf{V}_k(x) = \frac{1}{\Lambda}\int_0^\Lambda \mathbf{v}(x,t) e^{-2\pi kit/\Lambda} dt, \quad P_k(x) = \frac{1}{\Lambda}\int_0^\Lambda p(x,t) e^{-2\pi kit/\Lambda} dt.$$

Similar representation holds for the displacement vector $\mathbf{u}$.
These coefficients satisfy the following time independent problem

$$i\omega\mathbf{V} - \nu\Delta\mathbf{V} + \nabla P = 0 \;\text{ and }\; \nabla\cdot\mathbf{V} = 0 \;\text{ in } \mathcal{C},$$

with the Dirichlet boundary condition

$$\mathbf{V} = 0 \;\text{ on } \Gamma$$

or with elasticity conditions

$$D(\kappa(s), -\partial_s, -\partial_z)^T\,\overline{Q}(s)D(\kappa(s), \partial_s, \partial_z)\mathbf{U}(s,z) - \overline{\rho}(s)\omega^2\mathbf{U}(s,z)$$
$$+K\mathbf{U} + \rho_b\widehat{\mathcal{F}}(s,z) = 0,$$

$$\mathbf{V} = i\omega\mathbf{U} \;\text{ on } \Gamma\times\mathbb{R}$$

and $\widehat{\mathcal{F}} = (\widehat{\mathcal{F}}_n, \widehat{\mathcal{F}}_s, \widehat{\mathcal{F}}_z)$,

$$\widehat{\mathcal{F}}_n = -P + 2\nu\partial_n V_n, ...$$

$$\frac{\nu}{2}\int_0^T\int_\Omega \sum \varepsilon_{jk}^2 dxdt + \frac{1}{2}\int_\Omega |\mathbf{v}|^2 dx|_{t=T}$$
$$+\frac{1}{2\rho_b}\Big(a(\mathbf{u},\mathbf{u}) + \int_\Gamma \overline{\rho}|\partial_t\mathbf{u}|^2 dS_\Gamma\Big)|_{t=T} =$$
$$\frac{1}{2}\int_\Omega |\mathbf{v}|^2 dx|_{t=0}$$
$$+\frac{1}{2\rho_b}\Big(a(\mathbf{u},\mathbf{u}) + \int_\Gamma \overline{\rho}|\partial_t\mathbf{u}|^2 dS_\Gamma\Big)|_{t=0}$$

where

$$a(\mathbf{u},\mathbf{u}) = \int_\Gamma \Big(QD(\kappa,\partial_s,\partial_z)\mathbf{u}, D(\kappa,\partial_s,\partial_z)\mathbf{u}\Big)dS_\Gamma$$

Compared with the classical works of J.R. Womersley, our formulation of problem has much in common, both of them involve momentless shell theory for modeling the elastic wall. In Wormerley's works, axisymmetric pulsative blood flow in a vessel with circular isotropic elastic wall is found as a perturbation of the steady Poisseulle flow. Apart from inessential generalizations like arbitrary shape of vessel's cross-section and orthotropic wall, the main difference of our paper is as follows.

First, Wormsley's model does not take into account the reaction of the vessel wall to the radial (normal) component of the force acting on the wall from the side of the liquid. Within the momentless shell theory, elastic walls cannot resist the normal force in the case of weak reaction of the material on stretching and/or very small curvature of the cross-section. Both the cases are meaningful for diseased vessels due to atherosclerotic sequelae or aneurysm swelling but for healthy vessels, inflation of elastic wall is suppressed precisely by its stretching stress. Mathematically, the observed difference between the models in question results in the important fact that our model support standard Greens formula and it has a energy integral.

Second, we include the coefficient $K(s)$ describing the reaction of the surrounding cell material on deformation of the wall. Consider oscillations in the blood circulatory system due to the beating of the human heart. There are different factors which influence on oscillations of the blood flow and stresses and strains of the vessels wall. Elasticity of walls and interaction of walls with surrounded tissue contribute to the laminar part of the flow and low stresses and strains of the wall. Our result shows that all periodic flows are laminar and directed along the axis of cylinder in the case of elastic boundary.

One dimensional problem

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Thank you!

Questions?

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