A spectral approach to transverse linear instability of line periodic water waves

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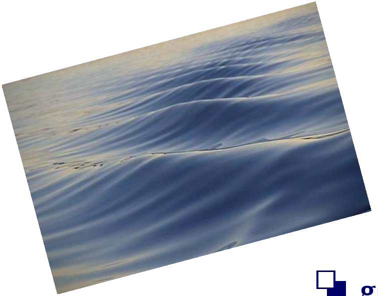
Periodic water waves

Water-wave problem



gravity-capillary waves

- *three-dimensional inviscid fluid layer*
- *constant density*
- *gravity and surface tension*
- *irrotational flow*

Euler equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \textit{for} \quad 0 < y < 1 + \eta$$

$$\phi_y = 0 \quad \textit{on} \quad y = 0$$

$$\phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \textit{on} \quad y = 1 + \eta$$

$$\phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \alpha\eta - \beta\,\mathcal{K} = 0 \quad \textit{on} \quad y = 1 + \eta$$

- **velocity potential** ϕ; **free surface** $h + \eta$
- mean curvature $\mathcal{K} = \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x + \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z$
- **parameters** ρ, g, σ, h

$$\boxed{\alpha = \frac{gh}{c^2}, \quad \beta = \frac{\sigma}{\rho h c^2}}$$

Euler equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \textit{for} \quad 0 < y < 1 + \eta$$

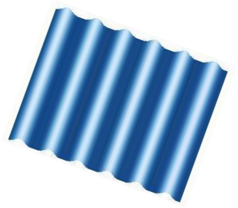
$$\phi_y = 0 \quad \textit{on} \quad y = 0$$

$$\phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \textit{on} \quad y = 1 + \eta$$

$$\phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \alpha\eta - \beta\,\mathcal{K} = 0 \quad \textit{on} \quad y = 1 + \eta$$

- **very rich dynamics**
  - symmetries, Hamiltonian structures
  - many particular solutions
- **difficulties**
  - variable domain (free surface)
  - nonlinear boundary conditions

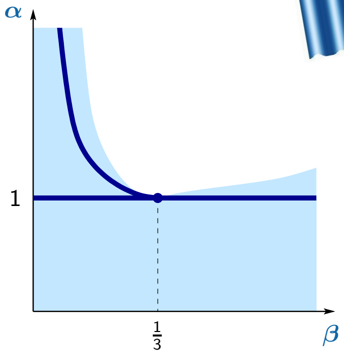
Focus on ...

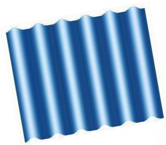


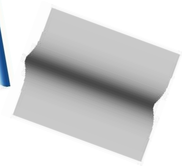
- traveling periodic 2D waves
- transverse linear instability
- analytical results
- Euler equations / long-wave models

2D PERIODIC TRAVELING WAVES

EXISTENCE RESULTS





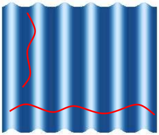


[Nekrasov, Levi-Civita, Struik, Lavrentiev, Friedrichs & Hyers, ... Amick, Kirchgässner, Iooss, Buffoni, Groves, Toland, Lombardi, Sun, ...]

Transverse linear (in)stability

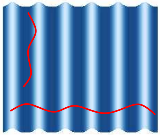
3D PERTURBATIONS

- *perturbations depend on both horizontal coordinates*



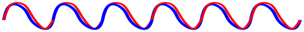
  - **transverse direction z:** *bounded (periodic)*
  - **longitudinal direction x:**

- *perturbations depend on both horizontal coordinates*

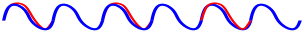


- **transverse direction z:** *bounded (periodic)*
- **longitudinal direction x:**

  co-periodic perturbations



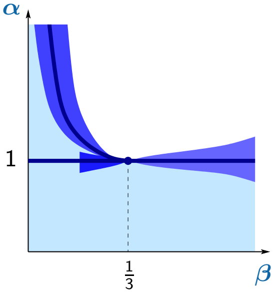
  subharmonic perturbations



  localized/bounded perturbations

- **long-wave models:** *transverse linear instability*

  perturbations: co-periodic in x, periodic in z



long-wave models: *transverse linear instability*

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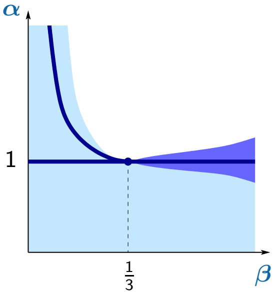

- **large surface tension:**

  Kadomtsev-Petviashvili I

$$u_{xt} = (u_{xx} + u + u^2)_{xx} - u_{yy}$$

[Johnson & Zumbrun (2010), H. (2011), Hakkaev, Stanislavova & Stefanov (2012), Rousset & Tzvetkov (2012), . . .]

Euler equations **[H. (2015)]**

- **long-wave models:** *transverse linear instability*

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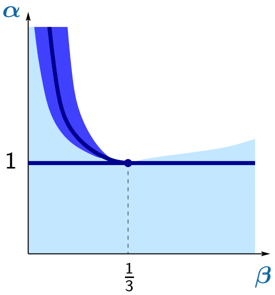


- **weak surface tension:**

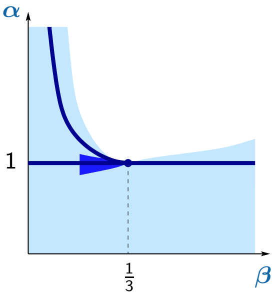
  Davey-Stewartson (E-E, focusing)

$$iA_t + A_{xx} + A_{yy} + (\gamma_1|A|^2 + \gamma_2\phi_x)A = 0$$
$$\gamma_3\phi_{xx} + \phi_{yy} - \gamma_3(|A|^2)_x = 0$$

[Godey (2016)]

- **long-wave models:** *transverse linear instability*

  perturbations: co-periodic in x, periodic in z



- **critical surface tension:**

  5th order KP equation

$$\boxed{u_{xt} = (u_{xxxx} + u_{xx} + u^2)_{xx} + u_{yy}}$$

[H. & Wahlén (2017)]

A common approach

[Godey (2016), Groves, Sun, & Wahlén (2016),

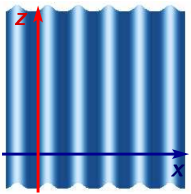
. . . Groves, H., & Sun (2001)]

Transverse instability problem

Transverse spatial dynamics

$$U_z = DU_t + F(U)$$

- $U(x, z, t)$, D *linear operator*, F *nonlinear map*
- **a periodic wave $U_*(x)$ is an equilibrium:** $F(U_*) = 0$



Transverse linear instability

Transverse spatial dynamics

$$\boxed{U_z = DU_t + F(U)}$$

$U_*(x)$ is **transversely linearly unstable** if the linearized system

$$\boxed{U_z = DU_t + \mathcal{L}U}, \quad \boxed{\mathcal{L} = F'(U_*)}$$

possesses a solution of the form $\boxed{U(x,z,t) = \mathrm{e}^{\lambda t} V_\lambda(x,z)}$

with $\lambda \in \mathbb{C}$, $Re\lambda > 0$, V_λ **bounded function.**

General instability criterion

Hypotheses

1. the **system** $\boxed{U_z = DU_t + F(U)}$ is **reversible**;
2. the **linear operator** $\boxed{\mathcal{L} = F'(U_*)}$ possesses **a pair of simple purely imaginary eigenvalues** $\pm i\kappa_*$;
3. the operators $\mathbf{D}$ and $\mathcal{L}$ are closed in $\mathcal{X}$ with $D(\mathcal{L}) \subset D(\mathbf{D})$;

THEOREM

1. For any $\lambda \in \mathbb{R}$ sufficiently small, the linearized system

$$\boxed{U_z = DU_t + \mathcal{L}U}$$

possesses a solution of the form $\boxed{U(\cdot, z, t) = e^{\lambda t} V_\lambda(\cdot, z)}$

with $V_\lambda(\cdot, z) \in D(\mathcal{L})$ **a periodic function in** z.

2. U_* **is transversely linearly unstable**.

[Godey (2016)]

Proof of 1.

- look for **solutions periodic in z of**

$$\boxed{U_z = (\lambda D + \mathcal{L})U}$$

- **the linear operator $\boxed{\lambda D + \mathcal{L}}$ possesses two complex conjugated purely imaginary eigenvalues**

PROOF OF 1.

- look for **solutions periodic in z of**

$$\boxed{U_z = (\lambda D + \mathcal{L})U}$$

- **the linear operator $\boxed{\lambda D + \mathcal{L}}$ possesses two complex conjugated purely imaginary eigenvalues**
  - $\mathcal{L}$ *possesses two simple eigenvalues* $\pm \mathbf{i}\kappa_*$;
  - $\lambda D + \mathcal{L}$ *is a small relatively bounded perturbation of* $\mathcal{L}$, *for small and real* λ;
  - *the spectrum of* $\lambda D + \mathcal{L}$ *is symmetric with respect to the imaginary axis* (due to reversibility) *and to the real axis* (since λ is real).

□

Euler equations

[H., T. Truong & E. Wahlén (2021)]

Hamiltonian formulation of the 3D problem:

$$\boxed{U_z = DU_t + F(U)}$$

- $U = (\eta, \omega, \phi, \xi)$ and $Y = y/(1+\eta)$
- boundary conditions

$$\boxed{\phi_y = b(U)_t + g(U)} \quad \textit{on} \;\; y = 0, 1$$

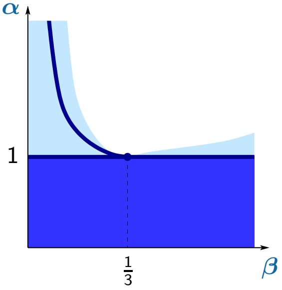
- the system is reversible

[Groves, H., & Sun (2001), Groves, Sun, & Wahlén (2016)]

[Groves & Mielke (2001)]

A one-parameter family of symmetric periodic waves

$\boldsymbol{\beta > 0,\ \alpha \in (0,1)}$ **linearly transversely unstable**



$$\eta(x) = \eta_\varepsilon(k_\varepsilon x),\ \phi(x,y) = \phi_\varepsilon(k_\varepsilon x, y)$$

- $\eta_\varepsilon, \phi_\varepsilon$ *are* 2π*-periodic in* $X = k_\varepsilon x$
- $k_\varepsilon = k_* + k_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$

  $\eta_\varepsilon = \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3)$

  $\phi_\varepsilon = \varepsilon\phi_1 + \varepsilon^2\phi_2 + O(\varepsilon^3)$
- *explicit formulas for* k_*, η_1, ϕ_1*,*

  and then also for k_2, η_2, ϕ_2*, ...*

Linearized system

- **linearized system** *(after rescaling $X = k_\varepsilon x$)*

$$\boxed{U_z = DU_t + DF(u_\varepsilon)U}$$

  - **boundary conditions**

$$\boxed{\phi_y = Db(u_\varepsilon)U_t + Dg(u_\varepsilon)U} \quad \text{on} \quad y = 0, 1$$

LINEARIZED SYSTEM

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- **linear operator** $\boxed{\mathcal{L}_\varepsilon := DF(u_\varepsilon)}$

  - **boundary conditions**

$$\boxed{\phi_y = Dg(u_\varepsilon)U} \quad \text{on} \quad y = 0, 1$$

  - space of 2π-periodic functions

$$\boxed{\mathcal{X} = H^1(0, 2\pi) \times L^2(0, 2\pi) \times H^1((0, 2\pi) \times (0, 1)) \times L^2((0, 2\pi) \times (0, 1))}$$

Linear operator $\mathcal{L}_\varepsilon$

$$\boxed{\mathcal{L}_\varepsilon = \mathcal{L}_0 + \mathcal{L}_\varepsilon^1} \quad \mathcal{L}_0 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{\omega}{\beta} \\ -k_\varepsilon^2 \beta \eta_{xx} + \alpha\eta - k_\varepsilon \phi_x|_{y=1} \\ \xi \\ -k_\varepsilon^2 \phi_{xx} - \phi_{yy} \end{pmatrix}, \ \mathcal{L}_\varepsilon^1 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ G_1 \\ G_2 \end{pmatrix}$$

$$g_1 = \frac{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{1/2}}{\beta}\left(\omega + \frac{1}{1+\eta_\varepsilon}\int_0^1 y\phi_{\varepsilon y}\xi\, dy\right) - \frac{\omega}{\beta}$$

$$\begin{aligned} g_2 = & \int_0^1 \left\{ k_\varepsilon^2 \phi_{\varepsilon x}\phi_x - \frac{\phi_{\varepsilon y}\phi_y}{(1+\eta_\varepsilon)^2} + \frac{\phi_{\varepsilon y}^2 \eta}{(1+\varepsilon\eta_\varepsilon)^3} - \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x}^2 \phi_{\varepsilon y}\phi_y}{(1+\eta_\varepsilon)^2} - \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x}\phi_{\varepsilon y}^2 \eta_x}{(1+\eta_\varepsilon)^2} + \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x}^2 \phi_{\varepsilon y}^2 \eta}{(1+\eta_\varepsilon)^3} \right. \\ & \left. + \left[k_\varepsilon^2 y\phi_{\varepsilon y}\phi_x + k_\varepsilon^2 y\phi_{\varepsilon x}\phi_y - \frac{2k_\varepsilon^2 y^2 \eta_{\varepsilon x}\phi_{\varepsilon y}\phi_y}{1+\eta_\varepsilon} - \frac{k_\varepsilon^2 y^2 \phi_{\varepsilon y}^2 \eta_x}{1+\eta_\varepsilon} + \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x}\phi_{\varepsilon y}^2 \eta}{(1+\eta_\varepsilon)^2} \right]_x \right\} dy \\ & + k_\varepsilon^2 \beta \eta_{xx} - k_\varepsilon^2 \beta \left[\frac{\eta_x}{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{3/2}} \right]_x \end{aligned}$$

$$G_1 = -\frac{\eta_\varepsilon \xi}{1+\eta_\varepsilon} + \frac{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{1/2}}{\beta(1+\eta_\varepsilon)}\left(\omega + \frac{1}{1+\eta_\varepsilon}\int_0^1 y\phi_{\varepsilon y}\xi\, dy\right) y\phi_{\varepsilon y}$$

$$\begin{aligned} G_2 = & \left[\frac{\eta_\varepsilon \phi}{(1+\eta_\varepsilon)} + \frac{\phi_\varepsilon \eta}{(1+\eta_\varepsilon)^2} \right]_{yy} - k_\varepsilon^2 [\eta_\varepsilon \phi_x + \phi_{\varepsilon x}\eta - y\phi_{\varepsilon y}\eta_x - y\eta_{\varepsilon x}\phi_y]_x \\ & + k_\varepsilon^2 \left[y\eta_{\varepsilon x}\phi_x + y\phi_{\varepsilon x}\eta_x + \frac{y^2\eta_{\varepsilon x}^2 \phi_{\varepsilon y}\eta}{(1+\eta_\varepsilon)^2} - \frac{y^2 \eta_{\varepsilon x}^2 \phi_y}{1+\eta_\varepsilon} - \frac{2y^2 \eta_{\varepsilon x}\phi_{\varepsilon y} y \eta_x}{1+\eta_\varepsilon} \right]_y \end{aligned}$$

Check hypotheses

- **Main step:** *show that $\mathcal{L}_\varepsilon$ possesses a pair of simple eigenvalues $\pm i\ell_\varepsilon$, for ε sufficiently small.*
  - *operator with compact resolvent $\longrightarrow$ pure point spectrum*

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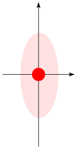
  - **spectrum of $\mathcal{L}_0$:** *only one purely imaginary eigenvalue*



- *eigenvalue* 0 *with geometric multiplicity* 3 *and algebraic multiplicity* 6

Check hypotheses

Main step: *show that $\mathcal{L}_\varepsilon$ possesses a pair of simple eigenvalues $\pm i\ell_\varepsilon$, for ε sufficiently small.*

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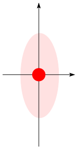
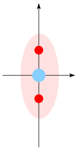


- *eigenvalue* 0 *with geometric multiplicity* 3 *and algebraic multiplicity* 6

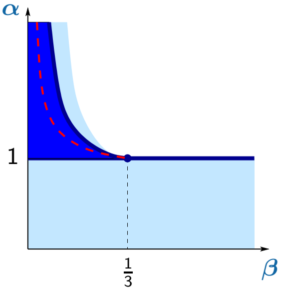
- *use a* **spectral decomposition and perturbation arguments** *to prove that $\mathcal{L}_\varepsilon$ has the eigenvalues:*



- *eigenvalue* 0 *with geometric multiplicity* 2 *and algebraic multiplicity* 4
- *a pair of simple eigenvalues $\pm i\ell_\varepsilon$*

Other 2d periodic waves

- **Two one-parameter families of symmetric periodic waves:** *same arguments and computations ...*



- *wavenumbers close to $\kappa_1 < \kappa_2$:*

  $$k_{\varepsilon,\kappa_1} = \kappa_1 + \mathcal{O}(\varepsilon^2)$$

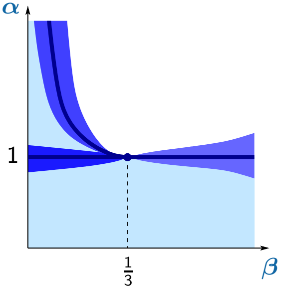
  $$k_{\varepsilon,\kappa_2} = \kappa_2 + \mathcal{O}(\varepsilon^2)$$

- *to the right of the red curve:* **both are transversely unstable**

- *to the left of the red curve:* **the family with wavenumbers k_{ε,κ_2} is transversely unstable**

Other 2d periodic waves

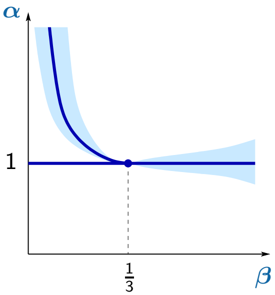
Close to the bifurcations curves:



- *same type of arguments ...*
- *the multiplicity of the eigenvalue* 0 *increases ... more involved computations ...*

Some related problems

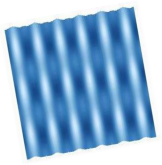
Solitary waves

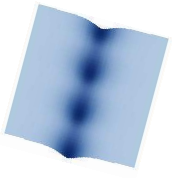


- **strong surface tension**
  - linear transverse instability
    [Groves, H. & Sun (2001), Pego & Sun (2004)]
  - nonlinear transverse instability
    [Rousset & Tzvetkov (2011)]
- **weak surface tension**
  - linear transverse instability
    [Groves, Sun, & Wahlén (2016)]

Dimension-breaking bifurcations

- existence of **transversely periodically modulated waves**





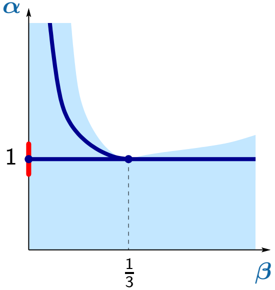
- **periodic waves:** large surface tension

  [H. (2015)]

- **solitary waves:** large and weak surface tension

  [Groves, H., & Sun (2001), Groves, Sun, & Wahlén (2016)]

Gravity waves



- model equation:

  Kadomtsev-Petviashvili II equation

$$u_{xt} = (u_{xx} + u + u^2)_{xx} + u_{yy}$$

- prediction: **transverse stability of periodic/solitary waves**

  [H., Li & Pelinovsky (2017)]

  [Mizumachi & Tzvetkov (2012), Mizumachi (2015)]

30